Solutions of Homework 7: CS321

Please show all steps in your work. Please be reminded that you should do your homework independently.

1. (10 points) The line is $y = ax + b$. If the line passes through $(x^*, y^*)$, we will have $y^* = ax^* + b$.

For the least squares method,

$$
\phi(a, b) = \sum_{i=0}^{m} (ax_i + b - y_i)^2,
$$

which gives the normal equations

$$
\left( \sum_{i=0}^{m} x_i^2 \right) a + \left( \sum_{i=0}^{m} x_i \right) b = \sum_{i=0}^{m} y_i x_i
$$

$$
\left( \sum_{i=0}^{m} x_i \right) a + (m+1)b = \sum_{i=0}^{m} y_i
$$

Look at the 2nd equation and dividing $(m+1)$ on both sides:

$$
\frac{1}{m+1} \left( \sum_{i=0}^{m} x_i \right) a + b = \frac{1}{m+1} \sum_{i=0}^{m} y_i
$$

Note that

$$
x^* = \frac{1}{m+1} \sum_{i=0}^{m} x_i, \quad y^* = \frac{1}{m+1} \sum_{i=0}^{m} y_i.
$$

The result follows immediately.

2. (10 points) Let

$$
\phi(a, b) = \sum_{k=0}^{m} (a x_k^2 + b - y_k)^2
$$

We have

$$
\phi(a, b) = (a + b - 3.1)^2 + (b - 0.9)^2 + (a + b - 2.9)^2.
$$

Set

$$
0 = \frac{\partial \phi}{\partial a} = 2(a + b - 3.1) + 2(a + b - 2.9)
$$

$$
0 = \frac{\partial \phi}{\partial b} = 2(a + b - 3.1) + 2(b - 0.9) + 2(a + b - 2.9)
$$

So we have

$$
2a + 2b = 6
$$

$$
2a + 3b = 6.9
$$

which yields $a = 2.1$ and $b = 0.9$. So the equation is $y = 2.1x^2 + 0.9$. 
3. (10 points)

\[ \phi(c) = \sum_{k=0}^{m} [f(x_k) - ce^{x_k}]^2 \]

Taking derivative with respect to \( c \) and setting it to 0, we have

\[ \frac{\partial \phi}{\partial c} = -2 \sum_{k=0}^{m} [f(x_k) - ce^{x_k}]e^{x_k} = 0 \]

Solving for \( c \), the result is

\[ c = \left[ \sum_{k=0}^{m} e^{x_k}f(x_k) \right] / \left[ \sum_{k=0}^{m} e^{2x_k} \right] \]

4. (10 points) Let

\[ \phi(a, b) = \sum_{k=0}^{m} (ax_k + b - y_k)^2 \]

We have

\[ \phi(a, b) = (a + b)^2 + (a + b - 1)^2 + (3a + b - 1)^2 + (4a + b - 2)^2 \]

Taking partial derivatives and setting them to 0,

\[ 0 = \frac{\partial \phi}{\partial a} = 2(a + b) + 2(a + b - 1) + 2(3a + b - 1)3 + 2(4a + b - 2)4 \]
\[ 0 = \frac{\partial \phi}{\partial b} = 2(a + b) + 2(a + b - 1) + 2(3a + b - 1) + 2(4a + b - 2) \]

which is simplified to

\[ 54a + 18b = 24 \]
\[ 18a + 8b = 8 \]

The solution is \( a = \frac{4}{9}, b = 0 \), and the linear equation is

\[ y = \frac{4}{9}x \]

5. (10 points) Let

\[ w_k = c_k + 3xw_{k+1} + 2w_{k+2}, \quad w_{n+2} = w_{n+1} = 0 \]

We have

\[ f(x) = \sum_{k=0}^{n} c_k g_k = \sum_{k=0}^{n} w_k g_k - 3x \sum_{k=0}^{n} w_{k+1} g_k - 2 \sum_{k=0}^{n} w_{k+2} g_k \]
\[ = \sum_{k=0}^{n} w_k g_k - 3x \sum_{k=1}^{n} w_k g_{k-1} - 3xw_{n+1} g_n - 2 \sum_{k=2}^{n} w_k g_{k-2} - w_{n+1} g_{n-1} - w_{n+2} g_n \]
\[ = w_0 g_0 + w_1 g_1 - 3xw_1 g_0 + \sum_{k=2}^{n} w_k (g_k - 3xg_{k-1} - 2g_{k-2}) \]
\[ = w_0 - w_1 (1 + 2x) \]
6. (10 points) Let

\[ \phi(a, b) = \sum_{k=0}^{m} (a e^{x_k^2} + b x_k^3 - y_k)^2 \]

So, we have

\[ \phi(a, b) = (ae - b)^2 + (a - 1)^2 + (ae + b - 2)^2 \]

Set

\[ 0 = \frac{\partial \phi}{\partial a} = 2(ae - b)e + 2(a - 1) + 2(ae + b - 2)e \]

\[ 0 = \frac{\partial \phi}{\partial b} = -2(ae - b) + 2(ae + b - 2) \]

This leads to the following system

\[ ae^2 - be + 1 - 1 + ae^2 + be - 2e = 0 \]
\[ -ae + b + ae + b - 2 = 0 \]

Solving the above system for \( a \) and \( b \), we have \( a = (1 + 2e)/(1 + 2e^2) \) and \( b = 1 \). So the equation is

\[ y = \frac{1 + 2e}{1 + 2e^2} e^{x^2} + x^3 \]