Sequence Semantics for Doxastic Logic

W. Marek [*][†] and M. Suchenek [‡]

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Abstract: In this paper we investigate sequence semantics for the modal logic of belief. This semantics differs from the previously considered in this that it allows flexibility in dealing with beliefs- those does not need to be closed under provability. We prove completeness theorem for such logic (with respect to this semantics) and show that stable theories correspond to constant models.

[*] Department of Computer Science, University of Kentucky, Lexington, KY 40506.

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[‡] Department of Computer Science, Wichita State University, Wichita, KS, 44444

1 Introduction

2 Sequence semantics

In this section we introduce a semantics for the modal language with the single modal operator S. This semantics is not based on the concept of possible world- like Kripke semantics for modal logic but rather on the explicit listings of statements "believed to an extent". Hence we have, informally speaking, collections of sentences objectively true (in the opinion of the agent), then the collection of statements that the agent finds plausible, then those which she believes to plausible to be plausible and so on. Hence such semantics involves infinitely many sets of formulas of the underlying language L. Initially, we do not require any committement on the part of the agent. In particular collections Σ_n of the sentences "plausible in the opinion of the agent to the degree $1/n$" do not need to be closed under consequence. They even may be inconsistent. Of course, when we show that, in fact our semantics includes that of so-called stable sets of Stalnaker [Sta80] and Moore [Moo85] we set natural conditions which allow us to get the embedding result.

Our semantics, called sequence semantics, is defined in steps. First of all let us define our language. Hence let L be the language of propositional logic. The new language L_S is obtained as follows:

(a) Treating the formulas of L as atoms.

(b) introducing new set of logical connectives (which, informally, can be treated as those serving to discuss believability of formulas of L). The connectives of L_S are an 0-ary connective 0_S, an unary connective S and binary connective $\Rightarrow$.

By L_S we do *not* mean the closure of L (treated as atoms) by S and $\Rightarrow$ but its subset defined inductively as follows:

(1) If ϕ belongs to L then $S\phi$ belongs to L_S.

(2) $0_S \in L_S$.

(3) If ϕ and ψ belong to L_S then $\phi \Rightarrow \psi$ also belongs to L_S.

In particular formulas of L are not members of L_S.

Finally, S is an operation mapping $L \cup L_S$ into L_S, except that we define $S(0_S) = 0_S$.

Hence, in particular, for $\phi, \psi \in L$ string $\phi \Rightarrow S\psi$ is not an expression from L_S, whereas, for instance, formula $SS\phi \Rightarrow 0_S$ is in L_S.

The idea of such a language over L is to have all the formulas of L "encapsulated" in belief operator. In this fashion it is quite clear that we discuss properties of the believing of these formulas.

The structures serving as models for our language are sequences of form: $\Sigma =< \Sigma_i : i \in \omega >$, of sets of formulas of L. The collection Mod, consists of all the structures of this form.

We define two operations on models; one of these is taking the "head" of the structure, another the the "tail" of it. Formally:

$$\Sigma! = \Sigma_0$$

$$\Sigma^\dagger =< \Sigma_{i+1} : i \in \omega >$$

We introduce now the notion of satisfaction for the formulas of L_S:

(Sat_1) $\Sigma \not\models 0_S$

(Sat_2) If $\phi \in L$ then: ($\Sigma \models S\phi$ iff $\phi \in \Sigma!$).

(Sat_3) If $\phi \notin L$, $\Sigma \models S\phi$ iff $\Sigma^\dagger \models \phi$

(Sat_4) $\Sigma \models \phi \Rightarrow \psi$ iff ($\Sigma \models \phi$ implies $\Sigma \models \psi$).

Finally, for $T \subseteq L_S$, $\Sigma \models T$ iff $\Sigma \models \psi$ for all $\psi \in T$. The notion of satisfaction determines the corresponding notion of entailment; $T \models \psi$

meaning that every model of T satisfies ψ.

One checks immediately the following fact:

$$\Sigma_n = \{\phi \in L : \Sigma \models S^{n+1}\phi\}$$

We intend to prove a certain completeness theorem for our semantics. Presently we introduce a collection of axioms, subsequently we prove completeness result.

First of all we introduce yet another operation which we call *normalization*. This operation, called f_S acts on $L \cup L_S$ and is defined as follows:

(N 1) $f_S(0_S) = 0_S$

(N 2) $f_S(\phi) = S\phi$, for $\phi \in L$

(N 3) $f_S(S\phi) = Sf_S(\phi)$, for $\phi \notin L$

(N 4) $f_S(\phi \Rightarrow \psi) = f_S(\phi) \Rightarrow f_S(\psi)$.

Let us look a little more closely at the operation f_S; Notice that $f_S(S\phi) = SS\phi$ for $\phi \in L$. On the other hand, $f_S(S(S\phi \Rightarrow S\psi))$ is $SSS\phi \Rightarrow SSS\psi$.

Now we are ready to specify the axiom system for our logic:

(Ax 1) All substitutions of formulas of L_S to the tautologies of propositional calculus (with $\Rightarrow$ and 0 as connectives)

(Ax 2) All formulas of form:

$$S\phi \Leftrightarrow f_S(\phi)$$

(Ax 3) Formulas $S\phi$ for ψ axioms.

The only rule of inference is Modus Ponens. The notion of provability ($\vdash$) is defined via existence of proof.

Let us see what is an effect of the axioms of group Ax 2. For instance let us look at the formula $f_S(S(S\phi \Rightarrow S\psi))$. According to Ax 2, this formula is equivalent to the effect to computing f_S on $SS\phi \Rightarrow SS\psi$. The latter formula is: $SSS\phi \Rightarrow SSS\psi$. The reader notices that we transformed our formula to one in which there is no occurrence of $\Rightarrow$ within the scope of S. In fact this is precisely the effect of the multiple application of the Ax 2 in general.

Hence, let a normal form formula be one in which there is no $\Rightarrow$ within a scope of S. Hence, for instance: $S(S\phi \Rightarrow S\phi) \Rightarrow S\theta$ is not a normal form formula, whereas $(SS\phi \Rightarrow S\psi) \Rightarrow S\theta$ is a normal form formula $(\phi, \psi, \theta \in L)$.

Simple facts on normal form formulas are immediately established.

Lemma 1 *(1) If $S\phi$ is a normal form formula, then for some $\theta \in L$, $\phi = S^k\theta$.*

(2) If ϕ is in normal form, then so is $f_S(\phi)$.

Crucial fact is:

Lemma 2 *(Normal Form) For every formula $\phi \in L_S$ there is a normal form formula $\psi \in L_S$ such that $\vdash_{Ax} \phi \equiv \psi$.*

Proof: By induction on complexity of formula ϕ.

1. (a) The cases of 0_S and $S\theta$ for $\theta \in L$ are obvious.

2. (b) The case of $\phi = \phi_1 \Rightarrow \phi_2$ where $\phi_1, \phi_2 \in L_S$ follows immediately by induction.

3. (c) The case of ϕ of form $S\theta$. Here we apply repeatedly method used in the example; Using axiom 2 we get an equivalent formula $f_S(\theta)$. The main connective of θ is either $\Rightarrow$ or S. In the former case, $\theta = \theta_1 \Rightarrow \theta_2$, we use inductive assumption for θ_1 and θ_2, finding equivalent formulas $\theta_1^{'}$ and $\theta_2^{'}$. Hence the formula $\theta_1^{'} \Rightarrow \theta_2^{'}$ is a normal form formula equivalent to $f_S(\theta)$, hence $S\theta$. In case the main connective of θ is S, we proceed as follows: $f_S(\theta^{'}) = Sf_S(\theta^{'})$. By axiom 2 again this formula is equivalent to $f_S(f_S(\theta^{'}))$. By inductive assumption $\theta^{'}$ possesses equivalent normal form formula. Applying f_S to it twice we get a normal form formula. This completes the argument. $\Box$.

One needs to see that a "shortcut", namely applying applying a formula $\theta \equiv \theta^{‘} \Rightarrow S\theta \equiv S\theta'$ is impossible. Namely, this formula is not provable in our system! Let us point that it is true for sequences Σ that are constant.

We are ready now to prove soundness of our axiomatization with respect to the sequence semantics.

Theorem 1 *(Soundness) If T is a theory in L_S, and $\phi \in L_S$, then $T \vdash_{Ax} \phi$ implies that T semantically entails ϕ.*

Proof: We need to prove that all our axioms are valid- which follows easily by induction, and that modus ponens leads from valid formulas to valid formula which is also obvious. $\Box$.

Now, we are able to prove the completeness theorem for our semantics using the usual consistency argument.

Theorem 2 *(Consistency) If T is consistent (i.e. $T \not\vdash 0_S$) then T possesses a model.*

Proof. Let T' be the collection of normal forms of formulas from T. If we prove that T' possesses a model then, by normal form theorem and soundness we are done.

Now, since T' consists of normal form and is consistent, there is a $0-1$ valuation V of formulas of form $S^{n+1}\phi$, $\phi \in L$ such that treating these formulas as atoms, $V(\phi) = 1$ for all $\phi \in T'$.

We define the structure Σ as follows: For $\phi \in L$, $\phi \in \Sigma_n$ if and only if $V(S^{n+1}(\phi) = 1$. Clearly, Σ is well defined. We just need to show that $\Sigma \models T'$. This, however is done easily by induction checking that:

$$V(\phi) = 1 \; iff \; S \models \phi$$

for all normal form formulas ϕ of L_S. $\Box$.

Corollary 1 (Completeness) *For all theories T of L_S and $\phi \in L_S$,*

$$T \vdash_A x\phi \text{ iff } T \models \phi$$

Now we shall introduce the sequence semantics for full language of modal logic (recall that the language L_S was only a subset of the modal language with the modal operator S).

In order to stress this difference we arbitrarily chose a new modal operator B and consider now full modal language L_B. In particular *formulas of L are not treated in L_B as atoms.* First we define an operation E which we call *encapsulation embedding.* This mapping of L_B into the language L_S is defined as follows:

1. (1) $E(\phi) = S\phi$ when $\phi \in L$,

2. (2) $E(0_B) = 0_S$,

3. (3) $E(B\phi) = SE(\phi)$,

4. (4) $E(\phi \Rightarrow \psi) = E(\phi) \Rightarrow E(\psi)$ providing that at least one of ϕ, ψ does not belong to L.

Now, the action of E looks at the first glance misterious but in fact its action is the following: We isolate in the formula ϕ *maximal* parts that do not contain the modal operator B and encapsulate those with the modal operator S. Then, B is changed to S.

Hence the operator $\Rightarrow$ has a double meaning: if it connects formulas which are objective, that is do not contain B then it is *not* interpreted as the operator $\Rightarrow$ of L_S. If, however it connects formulas (at least one of) which involves belief then it is interpreted as the operator of L_S. It may seem to look strange at the first glance but a moments reflection show that it is precisely in this fashion that the connectives are interpreted in related modes of reasoning (see [HM90], [Moo85]), for instance stable theories: The symbol $\Rightarrow$ in formula $Kp \Rightarrow (r \Rightarrow s)$ has, in fact two meanings: One of these is epistemic implication, it says that if p is in T (T is agent's knowledge base) then also $r \Rightarrow$ is in T; the other one does not tell us anything about relationship of r and s with respect to T.

We define now the satisfaction relation $\models^B$, between structures and formulas of L_B. 1j

1. $(Sat_B 1)$ If $\phi \in L$ then $\Sigma \models^B \phi$ iff $\phi \in \Sigma$!

2. $(Sat_B 2)$ $Not(\Sigma \models^B 0_B)$

3. $(Sat_B 3)$ If at least one of formulas $\phi, \psi \notin L$ then:

$$\Sigma \models^B \phi \Rightarrow \psi \text{ iff } \Sigma \models^B \phi \text{ implies } \Sigma \models^B \psi$$

4. $(Sat_B 4)$ $\Sigma \models^B B\phi$ iff $\Sigma^\dagger \models^B \phi$.

We have the following theorem:

Theorem 3 $\Sigma \models^B \phi$ iff $\Sigma^\dagger \models E(\phi)$

Proof: By induction on the number of connectives B in ϕ and subsequently nestings of $\Rightarrow$ within the scope of B.

References

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