Characterizing Pawlak's Approximation Operators

Victor W. Marek

Department of Computer Science
University of Kentucky
Lexington, KY 40506-0046, USA

To the memory of Zdzisław Pawlak,
in recognition of his friendship and guidance

Abstract. We investigate the operators associated with approximations in the rough set theory introduced by Pawlak in his [Paw82,MP84] and extensively studied by the Rough Set community [RS06]. We use universal algebra techniques to establish a natural characterization of operators associated with rough sets.

1 Introduction

The concept of *rough set determined by an equivalence relation* R has been introduced by Pawlak [Paw82,MP84] in his studies of data mining. It is a natural extension of a model of database introduced in [MP76] that treats records as objects which may be indiscernible in the language (i.e. the tables are bags, not sets, of records). Rough sets and a set of associated numerical measures allow for capturing various degrees of similarity of objects such as records, documents, or other data units. The point of departure of Pawlak was the realization that the descriptive languages are often inadequate to correctly describe concepts (i.e. – in set-theoretic terms – subsets of the domain). The express goal of rough sets was to operate in the following situation: we have a collection of objects X and some description language L. We have some collection of objects $Y \subseteq X$. We would like to describe Y in the language L. That is we would like to find a formula φ of L so that

$$Y = \{x \in X : \varphi[x]\}.$$

We call such sets Y *definable*. While usually the number of available definitions is infinite, even in the situation when X is finite, not every subset has to be definable. Yet another point, made in [MT99], is that a set Y may be definable in the language L but all the definitions are prohibitively large. In such circumstances we may want to find a smaller language L' where Y is *not* definable L', but the approximations are definable in L' by short formulas. This is certainly the case in various medical applications.

In his analysis, Pawlak observed that in the case of finite set X, there is a *largest* subset of Y that is definable, and a *least* superset of Y that is definable. There is a way to compute these largest and least definable subsets of X that are associated with Y. Specifically, this is done with the *indiscernibility relation* associated with the language L. In finite case, there is always a formula defining a *least definable set* containing a given object x. Let us call these sets *monads* (for the lack of better name, and for the fact that they resemble Leibniz monads). Then, it turns out that the largest definable set included in Y is the union of monads that are entirely included in Y, while the least definable set containing Y consists of those monads that have a nonempty intersection with Y. Abstracting from the existence of a specific language and its logical operations, Pawlak introduced the notion of *indiscernibility relation* in the set X. This is the equivalence relation R so that the monads are its cosets.

We believe that the guiding examples motivating Pawlak were standard medical terminologies such as SNOMED ([SN06]) or ICD-9 ([IC06]) and their inadequacies for description of classes of medical cases. It is worth mentioning that for many years Pawlak collaborated with physicians interested in Medical Informatics (needless to say, this started long before the term *Medical Informatics* were even coined). Pawlak was concerned with the fact that medical reasoning approximates the available data, often disregarding values of some attributes. As a result, it is often difficult, for a variety of reasons, to classify medical cases. If one treats a medical condition as an ideal set of cases and attempts to describe it within a concrete language of some terminology then all a physician can do is to produce a differential diagnosis. This leads, naturally, to lower and upper approximations of the classes of medical cases. While Pawlak's intuitions were motivated by his collaborations with practicing physicians, it turned out that the methodology of approximations and indiscernibility relations are a common phenomenon. We refer the reader to monographs and journals devoted to rough set theory ([RS06]) for further motivations.

Let us assume that the underlying set X is finite. Denoting by $\underline{R}(Y)$ and $\overline{R}(Y)$, respectively, the largest definable subset of Y and the least definable superset of Y, we get the desired approximation relationships

$$\underline{R}(Y) \subseteq Y \subseteq \overline{R}(Y).$$

The sets $\underline{R}(Y)$ and $\overline{R}(Y)$ provide collectively measure of adequacy of the underlying language to the task of describing Y. Moreover, by various statistical operations on those sets, and on other sets derived by set-theoretic means, we can analyze the properties of the set Y itself. For that reason we would like to know more about the sets $\overline{R}(Y)$ and $\underline{R}(Y)$. We would like to know what are possible operators of the form $\overline{R}(\cdot)$ and $\underline{R}(\cdot)$, and how those behave when R vary (i.e. when the language changes). These issues, to some extent were addressed in recent [GL06], but the review of the literature indicates that the Rough Sets community investigated a number of possible explanations for the rough set formalism by immersing it into various well-known mathematical areas. Those areas are all related to a variety of ways in which one can describe databases. We will list several different areas which were explored, although more could be mentioned. The references are, by necessity, incomplete. The first one is the idea of topological interpretation of approximations (already explored in [MP84]). This was for instance studied in [Yao96]. Another approach was to look at modal logic interpretations of rough sets (see for instance [YL96] and a series of papers by Orłowska and collaborators [DO01], and more generally [OS01]). One could think about approximations using Kleene [Kl67] three-valued logic as it was done in [MT99]. It is also possible to look at abstractions of rough sets via techniques of universal algebra. This last area explores Boolean algebra with operators [JT51,Jo91]. The rough set community investigated these connections, with a varying degree of generality, in [DO01,OS01,SI01]. Our contribution belong to this direction of research. We attempt to apply the techniques of universal algebra and in particular of [JT51,SI01] to find an *esthetically appealing* characterization of Pawlak's operators. In this quest our results find, indeed, a clean and interesting characterization of these operators. By necessity, some of the results discussed in this paper are known. After all operators over Boolean algebras have been introduced by Tarski and his collaborators over 50 years ago. For instance at least points (1)-(4) of Proposition 5 are known. The terms in which we characterize the Pawlak's operators are mostly known in the literature. The ones that we introduce and which (in conjunction with other properties) appear to be new are the following:

$$Y_1 \cap f(Y_2) \neq \emptyset \quad \text{if and only if} \quad Y_2 \cap f(Y_1) \neq \emptyset. \qquad \mathit{Exchange}$$

and

$$Y_1 \cup f(Y_2) \neq X \quad \text{if and only if} \quad Y_2 \cup f(Y_1) \neq X \qquad \mathit{Dual\ exchange}$$

As we will see, in addition to the well known properties of operators, these properties characterize the lower and upper Pawlak approximations, respectively.

Thus, in this paper, we prove four results that pertain to the explanation of Pawlak's approximation operators. First, we show a simple and elegant characterization of upper approximation. Much later we state but not prove the dual characterization (an indirect proof of this other property follow from duality considerations, and the point (5) of Proposition 5). We also prove the duality of exchange and dual exchange properties, and we show how one can introduce a structure of a complete lattice in upper approximation operators.

We believe that Pawlak, who believed in elegance of mathematical formulation of tools that are useful in practice, would enjoy the simplicity of our description of his operators.

2 Preliminaries

Given a set X and an equivalence (indiscernibility) relation R in X, we write $[x]_R$ for the R-*coset* of the element x in X, that is $\{y : xRy\}$. Given an equivalence relation R, the cosets of elements of X form a partition of X into nonempty blocks. We may drop the subscript R when R is determined by the context.

Let R be an equivalence relation in the set X. The relation R determines, for every set $Y \subseteq X$, two sets: the lower and upper R-bounds (also known as approximations) of Y. Specifically, following Pawlak [Paw82,MP84,Paw91] we define

$$\overline{R}(Y) = \{x \in X : [x] \cap X \neq \emptyset\}$$

and

$$\underline{R}(Y) = \{x \in X : [x] \subseteq X\}.$$

It is a simple consequence of the properties of equivalence relations and of De Morgan laws that for every subset Y of X, the complement of Y, $-Y$, has the following properties:

$$-\underline{R}(-Y) = \overline{R}(Y)$$

and

$$-\overline{R}(-Y) = \underline{R}(Y)$$

We now introduce the notion of an operator in a set X and introduce various classes of operators. Let X be a set. The set $\mathcal{P}(X)$ is the powerset of X, the collection of all subsets of X. Given a set X, by an *operator* in X we mean any function $f : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$. An operator f in the set X is *additive* if for all $Y_1, Y_2 \subseteq X$, $f(Y_1 \cup Y_2) = f(Y_1) \cup f(Y_2)$. An operator f in the set X is *multiplicative* if for all $Y_1, Y_2 \subseteq X$, $f(Y_1 \cap Y_2) = f(Y_1) \cap f(Y_2)$. An operator f in X is *progressive* if for all $Y \subseteq X$, $Y \subseteq f(Y)$. An operator f in X is *regressive* if for all $Y \subseteq X$, $f(Y) \subseteq Y$. An operator f in X is *idempotent* if for all $Y \subseteq X$, $f(f(Y)) = f(Y)$. An operator f in X *preserves empty set* if $f(\emptyset) = \emptyset$ (Operators preserving empty set are called *normal* in [JT51].) Finally, we say that an operator f in X *preserves unit* if $f(X) = X$.

All the properties of operators introduced above are pretty standard. Here are two properties (characteristic for our intended application) which are nonstandard. Let X be a set and let f be an operator in X. We say that f has an *exchange property* if for all $Y_1, Y_2 \subseteq X$,

$$Y_1 \cap f(Y_2) \neq \emptyset \quad \text{if and only if} \quad Y_2 \cap f(Y_1) \neq \emptyset.$$

This property of the operator will turn out to be crucial in our characterization of the upper approximation in Pawlak's rough sets.
Likewise, we we say that that f has a *dual exchange property* if for all $Y_1, Y_2 \subseteq X$

$$Y_1 \cup f(Y_2) \neq X \quad \text{if and only if} \quad Y_2 \cup f(Y_1) \neq X.$$

The dual exchange property will be used to characterize lower approximations of rough sets.

3 Characterizing $\overline{R}$

We now show the principal result of this note, the characterization of operations $\overline{R}$ for equivalence relations R (The characterization of lower approximations will follow from this result and the general facts regarding duality properties of operators.) We have the following result.

Proposition 1. *Let X be a finite set and let f be an operator in X. Then there exists an equivalence relation R such that $f = \overline{R}$ if and only if: f preserves empty set; f is additive; f is progressive; f is idempotent; and f has the exchange property.*

Proof: First, we need to show that whenever R is an equivalence relation in X then the operator $\overline{R}$ has the five properties listed above. The first four of these are pretty obvious; $\overline{R}$ preserves emptyset because when there is no element, then there is no coset. The additivity follows from the distributivity of existential quantifier with respect to disjunction, progressiveness follows from the fact that for all $x \in X$, $x \in [x]_R$, and the idempotence follows from the transitivity of the relation R. We will now show that the operator $\overline{R}$ possesses the exchange property. We observe that the exchange property is symmetric with respect to Y_1 and Y_2. Therefore all we need to prove is that whenever $Y_1 \cap \overline{R}(Y_2) \neq \emptyset$ then also $Y_2 \cap \overline{R}(Y_1) \neq \emptyset$. Let us reformulate slightly the statement $Y_1 \cap \overline{R}(Y_2) \neq \emptyset$. This statement is equivalent to the fact that there is an $x \in Y_1$, and an $y \in Y_2$ so that xRy. We now proceed as follows. Since $Y_1 \cap \overline{R}(Y_2) \neq \emptyset$, there is an element x that belongs to Y_1 and an element $y \in Y_2$ such that xRy. But then $[x]_R = [y]_R$, and so y is an element of Y_2 for which there is an element $x' \in Y_1$ so that yRx'. Namely x is that element x'. Therefore $Y_2 \cap \overline{R}(Y_1)$ is nonempty.

Now, let us assume that f is an operator in X, and that f has the five properties mentioned above, that is f preserves empty set, f is additive, f is progressive, f is idempotent, and that f has the exchange property. Then we need to construct an equivalence relation R_f so that f coincides with $\overline{R}$. Here is how we define relation R_f:

$$xR_fy \quad \text{if} \quad x \in f(\{y\}).$$

Our first task is to prove that, indeed, R_f is an equivalence relation in X. To see reflexiveness, let us observe that since f is progressive, for every x,

$$\{x\} \subseteq f(\{x\})$$

that is, $x \in f(\{x\})$. But this means that xR_fx, for every $x \in X$.
For the symmetry of R_f, let us assume xR_fy, that is $x \in f(\{y\})$. This means that

$$\{x\} \cap f(\{y\}) \neq \emptyset.$$

By the exchange property of f,

$$\{y\} \cap f(\{x\}) \neq \emptyset.$$

That is $y \in f(\{x\})$. In other words, yR_fx.
Finally, let us assume that x, y, z have the property that xR_fy and yR_fz. That is:

$$x \in f(\{y\}) \quad \text{and} \quad y \in f(\{z\}).$$

That is

$$\{x\} \subseteq f(\{y\}) \quad \text{and} \quad \{y\} \subseteq f(\{z\}).$$

From the second equality we have

$$\{y\} \cup f(\{z\}) = f(\{z\}).$$

By the additivity of f we have

$$f(\{y\}) \cup f(f(\{z\})) = f(f(\{z\})).$$

By idempotence of f we have, then

$$f(\{y\}) \cup f(\{z\}) = f(\{z\}).$$

This means that

$$f(\{y\}) \subseteq f(\{z\}).$$

But $x \in f(\{y\})$ and so $x \in f(\{z\})$, that is xR_fz, as desired.

To complete the proof of our assertion we need to prove that for all $Y \subseteq X$, $f(Y) = \overline{R_f}(Y)$. Our proof will use the fact that we deal with a finite set. We will comment on the dependence on this assumption later.

First, let us assume that $Y \subseteq X$, and that $x \in f(Y)$. Since X is finite, so is Y. Then

$$Y = \bigcup_{x \in Y} \{x\}.$$

Now, let us observe that since the operator f is additive, it is finitely additive that is it distributes with respect to finite unions. Thus:

$$f(Y) = \bigcup_{x \in Y} f(\{x\}).$$

This means that, since our assumption was that x belongs to $f(Y)$, for some $y \in Y$, $x \in f(\{y\})$. But then xR_fy for some $y \in Y$, that is $x \in \overline{R_f}(Y)$. In other words, for an arbitrary $Y \subseteq X$, $f(Y) \subseteq \overline{R_f}(Y)$.

Conversely, let us assume that $x \in \overline{R_f}(Y)$. Then, since we proved that R_f is an equivalence relation, for some $y \in Y$, xR_fy. That is, according to the definition of the relation R_f, $x \in f(\{y\})$. Next, we observe that f is monotone, that is $Y_1 \subseteq Y_2$ implies that $f(Y_1) \subseteq f(Y_2)$. Indeed, if $Y_1 \subseteq Y_2$ then $Y_1 \cup Y_2 = Y_2$, thus $f(Y_1 \cup Y_2) = f(Y_2)$ and by additivity $f(Y_1) \cup f(Y_2) = f(Y_2)$, that is $f(Y_1) \subseteq f(Y_2)$. Returning to the argument, since $y \in Y$, $\{y\} \subseteq Y$, and by our remark on monotonicity:

$$f(\{y\}) \subseteq f(Y).$$

This implies that $x \in f(Y)$ and since x was an arbitrary element of $\overline{R_f}(Y)$, $\overline{R_f}(Y) \subseteq f(Y)$. Thus we proved the other inclusion and since Y was the arbitrary subset of X, we proved that f and $\overline{R_f}$ coincide. $\square$

In the proof of our Proposition 1 we computed, out of the operator f, a relation R_f so that $f = \overline{R_f}$. But this relation is unintuitive (at least for non-specialists). We will now provide a more intuitive description of the same relation. Given an operator f, we define a relation S_f as follows:

$$xS_fy \quad \text{if} \quad \forall_{Y \subseteq X}(x \in f(Y) \Leftrightarrow y \in f(Y)).$$

We now have the following result.

Proposition 2. *If the operator f satisfies the conditions of Proposition 1, then $R_f = S_f$*

Proof: We need to prove two implications:
(a) $\forall_{x,y}(xR_fy \Rightarrow xS_fy)$, and
(b) $\forall_{x,y}(xS_fy \Rightarrow xR_fy)$
To show (a) let x, y be arbitrary elements of X, and let us assume xR_fy. Then, since R_f is symmetric, yR_fx, that is $y \in f(\{x\})$. It is sufficient to prove that for all subsets

Y of X, if $x \in f(Y)$ then $y \in f(Y)$ (the proof of the converse is very similar, except that we use the fact that $x \in f(\{y\})$). So, let $x \in f(Y)$. Then $\{x\} \subseteq f(Y)$, so, by monotonicity, $f(\{x\}) \subseteq f(f(Y)) = f(Y)$ (last equality uses idempotence of f). Thus, $f(\{x\}) \subseteq f(Y)$, and since $y \in f(\{x\})$, $y \in f(Y)$. Thus, taking into account the other implication, proved as discussed above, we proved that $xR_f y$ implies $xS_f y$.

Next, let us assume that $xS_f y$. That is,

$$\forall_{Y \subseteq X}(x \in f(Y) \Leftrightarrow y \in f(Y)).$$

We need to prove that $x \in f(\{y\})$. But $y \in f(\{y\})$, since for $Y = \{y\}$, $y \in Y$, and for every Y, $Y \subseteq f(Y)$ (f is progressive). But now specializing the above equivalence to $Y = \{y\}$, we find that $x \in f(\{y\})$, as desired. □

In the proof of Proposition 1 we used the assumption that X was a finite space. In fact, we could relax this assumption, but at a price. Recall that we assumed that the operator f was additive (i.e. $f(Y_1 \cup Y_2) = f(Y_1) \cup f(Y_2)$, for all subsets Y_1, Y_2 of X). In the case when X is finite we have for any family $\mathcal{X}$ of subsets of X

$$f(\bigcup \mathcal{X}) = \bigcup_{Y \in \mathcal{X}} f(Y).$$

This is easily proved by induction on the size of $\mathcal{X}$. Let us call an operator f *completely additive* if the equality

$$f(\bigcup \mathcal{X}) = \bigcup_{Y \in \mathcal{X}} f(Y).$$

holds for *every* family $\mathcal{X}$ of subsets of X. Under the assumption of complete additivity the assumption of finiteness can be eliminated.

4 Structure of the family of upper closure operators

We will now look at the situation when the set X has several different equivalence relations, that is several corresponding notions of rough sets. This is, actually, quite common situation; for instance we may have different medical nomenclature systems that are used to describe medical cases. In fact it is a well-known fact that the medical nomenclatures of various nations are not translatable.

We now face the question of the relationship between the different upper closure operators. Specifically, we may want to check the relationship between $\overline{R_1}$ and $\overline{R_2}$ given relations R_1 and R_2.

Proposition 3. *Let R_1, R_2 be two equivalence relations. Then*

$$R_1 \subseteq R_2 \quad \textit{if and only if} \quad \forall_{Y \subseteq X}(\overline{R_1}(Y) \subseteq \overline{R_2}(Y)).$$

Proof: First, let us assume that $R_1 \subseteq R_2$, and let Y be an arbitrary subset of X. We need to prove $\overline{R_1}(Y) \subseteq \overline{R_2}(Y)$. To this end, let $x \in \overline{R_1}(Y)$. Then there is an element $y \in Y$ such that xR_1y. But then xR_2y and so $x \in \overline{R_2}(Y)$.

Conversely, let us assume that for every Y, $\overline{R_1}(Y) \subseteq \overline{R_2}(Y)$. We want to prove that $R_1 \subseteq R_2$. Let us assume that xR_1y. Then $x \in \overline{R_1}(\{y\})$, thus $x \in \overline{R_2}(\{y\})$. In other words, there is some $y' \in \{y\}$ such that xR_2y'. But $\{y\}$ has unique element, y. Thus xR_2y, as desired. $\Box$

The structure of the family of all equivalence relations in a set X is well-known. Let $\langle EqR_X, \subseteq\rangle$ be the relational structure with EqR_X equal to the set of all equivalence relations in X, ordered by inclusion. Then $\langle EqR_X, \subseteq\rangle$ is a complete lattice (regardless whether X is finite or not) but $\langle EqR_X, \subseteq\rangle$ is not a distributive lattice, in general ([Ho93, p. 227]).

Proposition 3 allows us to transfer the properties of equivalence relations to operators. Let us define an *upper rough set operator* in the set X as any operator that preserves empty set, is completely additive (thus we no longer assume X to be finite), progressive, idempotent, and has the exchange property. We denote by $\mathcal{R}_X$ the set of all upper rough set operators in X, and $\preceq_X$ the *dominance relation* in $\mathcal{R}_X$ defined by

$$f \preceq g \quad \text{if} \quad \forall_{Y \subseteq X}(f(Y) \subseteq g(Y)).$$

Then applying our discussion of the lattice of equivalence relations in X to Proposition 3 we get the following fact.

Proposition 4. *The structure $\langle \mathcal{R}_X, \preceq\rangle$ is a poset. In fact $\langle \mathcal{R}_X, \preceq\rangle$ is a complete lattice, but in general not a distributive one.*

5 Duality

Let f be an operator in a set X. The *dual* of the operator f, f_d, is an operator defined by the following equality:

$$f_d(Y) = -f(-Y).$$

Here Y ranges over arbitrary subsets of X, $-Y = X \setminus Y$ is the complement operation. The dual operators are used in various places in mathematics and computer science. One example is the operator dual to van Emden-Kowalski operator T_P ([Do94, p. 83]).

While we defined the notion of dual operator in the Boolean lattice, $\langle \mathcal{P}(X), \subseteq\rangle$, as long as the lattice has a complement operation $-$, the notion of a dual operator can be defined. Moreover, if for all x, $--x = x$, then $(f_d)_d = f$. This is certainly the case in our application.

Now, let us assume that we are dealing with operators in a set X. We have the following fact.

Proposition 5. *Let X be a set and f an operator in X. Then:*

1. *The operator f preserves the empty set (unit) if and only if the operator f_d preserves the unit (empty set)*
2. *The operator f is progressive (regressive) if and only if the operator f_d is regressive (progressive)*
3. *The operator f is additive (multiplicative) if and only if the operator f_d is multiplicative (additive)*
4. *The operator f is idempotent if and only if the operator f_d is idempotent*

5. *The operator f possesses the exchange property (dual exchange property) if and only if the operator f_d possesses the dual exchange property (exchange property).*

Proof: The points (1)-(3) are entirely routine. To see the point (4), let us assume that the operator f is idempotent. Then for an arbitrary Y,

$$f_d(f_d(Y)) = -f(-f_d(Y)) = -f(--f(-Y)) = -f(f(-Y)) = -f(-Y) = f_d(Y).$$

The penultimate equality uses the idempotence of f. The other direction of (4) follows from the fact that $(f_d)_d = f$, and the argument above.
To see (5), we first assume that f has the exchange property. We prove that f_d has the dual exchange property. To this end we need to prove that for arbitrary $Y_1, Y_2 \subseteq X$,

$$Y_1 \cup f_d(Y_2) \neq X \quad \text{if and only if} \quad Y_2 \cup f_d(Y_1) \neq X.$$

Since this formula is symmetric with respect to Y_1 and Y_2, it is enough to prove the implication:

$$Y_1 \cup f_d(Y_2) \neq X \ \Rightarrow \ Y_2 \cup f_d(Y_1) \neq X.$$

So let us assume that $Y_1 \cup f_d(Y_2) \neq X$. Then, substituting $--Y_1$ for Y_1, and expanding the definition of f_d, we get:

$$(- - Y_1) \cup -f(-Y_2) \neq X$$

that is:

$$-(-Y_1 \cap f(-Y_2)) \neq X.$$

This is, of course, equivalent to:

$$-Y_1 \cap f(-Y_2) \neq \emptyset.$$

Since f has the exchange property,

$$-Y_2 \cap f(-Y_1) \neq \emptyset.$$

Thus we get:

$$-(-Y_2 \cap f(-Y_1)) \neq X,$$

which reduces to

$$Y_2 \cup -f(-Y_1) \neq X.$$

that is

$$Y_2 \cup f_d(Y_1) \neq X,$$

as desired. The proof of the other part of (5) namely that whenever f has the dual exchange property then f_d has the exchange property, is similar. □

Now, let us look at the familiar equality $\underline{R}(Y) = -\overline{R}(-Y)$. This, in the language of operators, says that for every equivalence relation R, the operator $\underline{R}$ is simply $\overline{R}_d$. So now we compare the characterization of the upper approximation by five conditions (Proposition 1) and the duality result above (Proposition 5). We get the following result.

Proposition 6. *Let X be a finite set and let f be an operator in X. Then there exists an equivalence relation R such that $f = \underline{R}$ if and only if: f preserves unit; f is multiplicative; f is regressive; f is idempotent; and f has the dual exchange property.*

Again, we can also study the family of all operators that have the five properties of operators characterizing lower approximation and introduce a complete, but non-distributive lattice structure in that set. That is, we can prove the result analogous to the Proposition 4.

6 Conclusions

Algebraic methods, whenever applicable, provide a clean foundations for an underlying subject. They abstract from unnecessary details, showing the properties that really matter. This is certainly the case in the area of rough sets. Our results confirm that, as observed by numerous authors [OS01,SI01] the theory of rough sets relates to the operators in lattices, a theory well-developed ([DP92, p. 86, ff.]) and with many deep results. Rough sets approximate elements of one lattice (Boolean lattice of all sets) with elements of a sublattice (of definable sets). Abstract approach to this idea of approximation and characterization of approximations in algebraic terms will only improve our understanding of the concept of rough set. We find it amazing that the ideas of Tarski (who certainly shied from applications) found its expression in Pawlak's, very applied, research.

Acknowledgments

This research was supported by the National Science Foundation under Grant 0325063. We are grateful to A. Skowron for directing us to a number of references. An anonymous referee pointed relations to several cited papers. We express to her/him gratitude for indicating the connections to the universal algebra research.

References

[DP92] Davey, B.A., and Priestley, H.A., *Introduction to Lattices and Order*, Cambridge University Press, 1992.

[Do94] K. Doets, *From Logic to Logic Programming*, MIT Press, 1994.

[DO01] I. Düntsch and E. Orłowska Beyond Modalities: Sufficiency and Mixed Algebras. Chapter 16 of [OS01], 2001.

[Ho93] W. Hodges, *Model Theory*. Cambridge University Press, 1993.

[IC06] System for medical diagnoses ICD-9, http://pmiconline.com.

[Jo91] B. Jonsson A Survey of Boolean Algebras with Oprators. In: *Algebras and Order*, pages 239–284. Kluwer, 1991.

[JT51] B. Jonsson and A. Tarski. Boolean Algebras with Operators. *American Journal of Mathematics* 73:891–939, 1951.

[Kl67] S. C. Kleene. *Introduction to Metamathematics*. North-Holland, 1967. Fifth reprint.

[GL06] Liu, G-L., The axiomatization of the Rough Set Upper Approximation Operations, *Fundamenta Informaticae* 69:331–342, 2006.

[MP76] W. Marek and Z. Pawlak. Information storage and retrieval systems, mathematical foundations, *Theoretical Computer Science* 1(4):331–354, 1976.

[MP84] W. Marek and Z. Pawlak. Rough sets and information systems, *Fundamenta Informaticae* 7(1):105–115, 1984.

[MT99] V.W. Marek and M. Truszczyński. Contributions to the Theory of Rough Sets, *Fundamenta Informaticae* 39(4):389–409, 1999.

[OS01] Ewa Orłowska and Andrzej Szałas. *Relational Methods for Computer Science Applications. Selected Papers from 4th International Seminar on Relational Methods in Logic, Algebra and Computer Science (RelMiCS'98)*, Studies in Fuzziness and Soft Computing 65, Physica-Verlag/Springer, 2001.

[Paw82] Z. Pawlak. Rough Sets, *International Journal of Computer and Information Sciences* 11:341–356, 1982.

[Paw91] Pawlak, Z., *Rough Sets – theoretical aspects of reasoning about data*, Kluwer, 1991.

[RS06] Rough Sets, a journal series of *Springer Lecture Notes in Computer Science*.

[SI98] E. SanJuan and L. Iturrioz. Duality and informational representability of some information algebras. pages 233-247, in: *Rough Sets in Knowledge Discovery, Methodology and Applications*, L. Polkowski and A. Skowron, (eds). Physica-Verlag, 1998.

[SI01] E. SanJuan and L. Iturrioz. An application of standard BAO theory to some abstract Information algebras. Chapter 12 of [OS01], 2001.

[SN06] System of medical terminology SNOMED, http://www.snomed.org.

[Yao96] Y.-Y. Yao. Two views of the theory of rough sets in finite universes. *International Journal of Approximate Reasoning* 15:291–317, 1996.

[YL96] Y.-Y. Yao and T.Y. Lin. Generalization of rough sets using modal logic, *Intelligent Automation and Soft Computing* 2:103–120, 1996.