On the number of minimal transversals in 3-uniform hypergraphs

Zbigniew Lonc*
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warsaw, Poland
zblonc@mini.pw.edu.pl

Mirosław Truszczyński
Department of Computer Science
University of Kentucky
Lexington, KY 40506-0046, USA
mirek@cs.uky.edu

June 29, 2007

Abstract

We prove that the number of minimal transversals (and also the number of maximal independent sets) in a 3-uniform hypergraph with n vertices is at most c^n, where $c \approx 1.6702$. The best known lower bound for this number, due to Tomescu, is ad^n, where $d = 10^{\frac{1}{5}} \approx 1.5849$ and a is a constant.

1 Introduction

An *independent set* in a graph is a set of vertices that contains no edge. An independent set is *maximal* if it is not a proper subset of any other independent set. In 1965 Moon and Moser [8] provided a complete answer to the following question raised by Erdős and Moser: "What is the maximum number $f(n)$ of maximal independent sets possible in a graph with n vertices, and which graphs have that many maximal independent sets?"

Moon and Moser proved that for every $n \geq 2$, the extremal graphs are the graphs whose every connected component is a triangle, except that if $n \mod 3 = 2$ one component is an edge and if $n \mod 3 = 1$ one component is K_4 or two components are edges. Thus,

$$f(n) = \begin{cases} 3^{n/3} & \text{if } n \mod 3 = 0 \\ 4 \times 3^{(n-4)/3} & \text{if } n \mod 3 = 1 \\ 2 \times 3^{(n-2)/3} & \text{if } n \mod 3 = 2. \end{cases}$$

There is a natural generalization of the problem solved by Moon and Moser. Let $\mathcal{G}$ be a fixed family of graphs. We ask: "What is the maximum number $f_{\mathcal{G}}(n)$ of maximal independent sets possible in a graph in $\mathcal{G}$ with n vertices, and which graphs in $\mathcal{G}$ have that many maximal

*corresponding author, fax (48) 22 625 7460, phone (48) 22 621 9312

independent sets?" Several authors considered this problem for some special classes $\mathcal{G}$ of graphs. Wilf [11] and independently Sagan [9] solved this problem when $\mathcal{G}$ is the family of trees. Griggs, Grinstead and Guichard [1] and independently Füredi [2] answered the question when $\mathcal{G}$ is the family of connected graphs. Hujter and Tuza [3] determined $f_{\mathcal{G}}(n)$ and found the corresponding extremal graphs when $\mathcal{G}$ is the family of triangle-free graphs.

It is also natural to ask an analogous question to the one posed by Erdős and Moser for hypergraphs. By a *hypergraph* we mean a finite family of finite sets. We refer to these sets as *edges* of the hypergraph. Given a hypergraph $\mathcal{H}$, we set $V(\mathcal{H}) = \bigcup \mathcal{H}$ and call elements of $V(\mathcal{H})$ the *vertices* of $\mathcal{H}$. A set of vertices in $\mathcal{H}$ is *independent* if it contains no edge of $\mathcal{H}$. An independent set in $\mathcal{H}$ is *maximal* if it is not a proper subset of any other independent set.

Here is a hypergraph analog of the question of Erdős and Moser: "Given a fixed family of hypergraphs $\mathcal{G}$, what is the maximum number $f_{\mathcal{G}}(n)$ of maximal independent sets possible in a hypergraph in $\mathcal{G}$ with n vertices, and which hypergraphs have that many maximal independent sets?"

This question is easy to answer when $\mathcal{G} = \mathcal{G}_{all}$ is the family of all hypergraphs. Let $\mathcal{S}_n$ be the hypergraph on n vertices whose edges are all sets of vertices of cardinality $\lfloor n/2 \rfloor + 1$. Clearly, a set $I \subseteq V(\mathcal{S}_n)$ is a maximal independent set in $\mathcal{S}_n$ if and only if $|I| = \lfloor n/2 \rfloor$. Hence $f_{\mathcal{G}_{all}}(n) \geq \binom{n}{\lfloor n/2 \rfloor}$. On the other hand if I_1, I_2 are two different maximal independent sets in some hypergraph then, obviously, $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$ so by Sperner's Lemma $f_{\mathcal{G}_{all}}(n) \leq \binom{n}{\lfloor n/2 \rfloor}$. Consequently, $f_{\mathcal{G}_{all}}(n) = \binom{n}{\lfloor n/2 \rfloor}$.

For a given $k \geq 2$, we denote by $\mathcal{C}_k$ the family of k-uniform hypergraphs, i.e. the family of hypergraphs for which every edge is a k-element set. Tomescu [10] raised the question of finding $f_{\mathcal{C}_k}(n)$ (for a fixed k) and determining the family of the corresponding extremal hypergraphs. Since $\mathcal{C}_2$ is the class of all graphs without isolated vertices, the result by Moon and Moser resolves the case $k = 2$ in the problem of Tomescu.

For $k > 2$, Tomescu gave a construction of hypergraphs providing a lower bound for $f_{\mathcal{C}_k}$. He conjectured that the actual value of $f_{\mathcal{C}_k}(n)$ is equal to this lower bound.

Here is the construction of Tomescu [10] in the case of $k = 3$. which is the case of our main interest in this paper. For simplicity, we assume that n is divisible by 5. Let $X_1, X_2, \ldots, X_{n/5}$ be pairwise disjoint 5-element sets. We denote by $\mathcal{P}_3(X_i)$, $i = 1, 2, \ldots, n/5$, the family of all 3-element subsets of X_i. We define the hypergraph

$$\mathcal{H}_n^3 = \bigcup_{i=1}^{n/5} \mathcal{P}_3(X_i).$$

Clearly, a set I of vertices of $\mathcal{H}_n^3$ is a maximal independent set in $\mathcal{H}_n^3$ if and only if the intersection of I with each of the sets X_i has 2 elements. Thus, there are $\binom{5}{2}^{n/5} = 10^{n/5} = d^n$, where $d = 10^{\frac{1}{5}} \approx 1.5849$, maximal independent sets in $\mathcal{H}_n^3$ and each of them has $2n/5$ elements. Consequently, $f_{\mathcal{C}_3}(n) \geq d^n$, for n divisible by 5. A simple corollary of this observation is that there is a constant a such that *for every* n, $f_{\mathcal{C}_3}(n) \geq ad^n$.

To the best of our knowledge the conjecture of Tomescu remains open and no nontrivial upper bound for the number $f_{\mathcal{C}_3}(n)$ is known. In this paper we show that $f_{\mathcal{C}_3}(n) \leq c^n$, where $c \approx 1.6702$. The exact value of c is of the form $1 + x^4$, where x is the root in $(0, 1]$ of the polynomial $f(x) = (x^4+1)^8 - (x^4+1)^7 + x^2(x^4+1)^4 + 2x(x^4+1)^2 + 1 - (x^4+1)^8 x^5$. In fact, we prove a stronger result that $f_{\mathcal{C}_{\leq 3}}(n) \leq c^n$, where $\mathcal{C}_{\leq 3}$ is the family of hypergraphs with every edge of cardinality at most 3.

We formulate and prove our results in terms of transversals of hypergraphs rather than in terms of independent sets in hypergraphs. For a hypergraph $\mathcal{H}$, a set $X \subseteq V(\mathcal{H})$ is a *transversal* of $\mathcal{H}$ if $X \cap E \neq \emptyset$ holds for every edge $E \in \mathcal{H}$. A transversal is *minimal* if none of its proper subsets is a transversal. One can readily verify that a set X is a maximal independent set in a hypergraph $\mathcal{H}$ if and only if $V(\mathcal{H}) \setminus X$ is a minimal transversal in $\mathcal{H}$. Therefore the number of maximal independent sets in any hypergraph $\mathcal{H}$ is equal to the number of minimal transversals in $\mathcal{H}$. Consequently, in particular, $f_{\mathcal{G}}(n)$ is also equal to the maximum number of minimal transversals possible in a hypergraph in $\mathcal{G}$ with n vertices. Similarly, the extremal hypergraphs with $f_{\mathcal{G}}(n)$ minimal transversals are the same as the extremal hypergraphs with $f_{\mathcal{G}}(n)$ maximal independent sets.

The problem of determining $f_{\mathcal{C}_3}(n)$ and finding hypergraphs in $\mathcal{C}_3$ maximizing the number of minimal transversals (equivalently maximal independent sets) is interesting by its own right. Nevertheless the motivation of the research presented in this paper also comes from some problems occurring in logic.

Let X be a set of n Boolean variables. By a *literal* we mean a variable x or its negation $\neg x$, where $x \in X$. A *clause over* X is a disjunction $\neg x_1 \vee \ldots \vee \neg x_s \vee y_1 \vee \ldots \vee y_t$, where $x_1, \ldots, x_s, y_1, \ldots, y_t \in X$. Finally, a *CNF theory over* X is a conjunction of clauses over X. A *truth valuation* for a CNF theory T is a function which assigns to every variable in X the logic value of **true** or **false**. A truth valuation *satisfies* T if it satisfies all the clauses in T. We say that a set $M \subseteq X$ is a *model* of a CNF theory T, if the truth valuation that assigns the value of **true** to all variables in M and the value of **false** to all variables in $X \setminus M$ satisfies the theory T. Clearly, for any CNF theory T, there is a one-to-one correspondence between truth valuations that satisfy T and models of T.

Minimal models of CNF theories play an important role in logic programming (see Lifschitz [5], McCarthy [7]). Therefore algorithms of generating all minimal models of CNF theories and counting them are of great interest (see Lonc and Truszczyński [6]).

When we restrict our attention to CNF theories in which no negation symbol occurs then the problem of generating and counting minimal models in such theories reduces to the problem of generating and counting transversals in some related hypergraphs. Indeed, let T be a CNF theory over a set of variables X in which no variable occurs negated and let $C_1, C_2, \ldots, C_m$ be the clauses in T. We define a hypergraph

$$\mathcal{H}(T) = \{E_i : \ E_i \text{ is the set of variables occurring in } C_i, \ i = 1, \ldots, m\}.$$

We observe that a set $M \subseteq X$ is a (minimal) model of T if and only if M is a (minimal) transversal in $\mathcal{H}(T)$. Thus, our results on minimal transversals in hypergraphs can be translated into results on minimal models in CNF theories in which no negation symbol occurs.

In the rest of the paper we describe the proof of our main result (Theorem 2.1). The argument uses a combinatorial lemma (Lemma 2.7) whose proof is long and requires a tedious case analysis. Therefore, we only sketch it in the main body of the paper and present all details in the appendix.

In this paper we write real numbers in the form $v.wxyz..$, where v, w, x, y, z are decimal digits in the decimal expansion of this real number.

2 Main result

We observe that our definitions do not allow for a hypergraph to have isolated vertices. That does not affect the generality of our considerations as isolated vertices are immaterial for properties of transversals.

The main result of our paper establishes an upper bound on the number of minimal transversals in a 3-uniform hypergraph.

Theorem 2.1 *Every 3-uniform hypergraph $\mathcal{H}$ with n vertices has at most $1.6701..^n$ minimal transversals.*

We derive Theorem 2.1 from a stronger result. We recall that $\mathcal{C}_{\leq 3}$ denotes the class of all hypergraphs with every edge of cardinality at most 3.

Theorem 2.2 *Every hypergraph $\mathcal{H} \in \mathcal{C}_{\leq 3}$ such that $|V(\mathcal{H})| = n$ has at most $1.6701..^n$ minimal transversals.*

In the remainder of this section, we will prove Theorem 2.2. The proof depends on a technical lemma. We outline the proof of the lemma here and provide the details of the proof in the appendix.

We start by introducing concepts and notation needed for the proof. Let V be a set. We call a pair $A = (A^+, A^-)$ of *disjoint* subsets $A^+, A^- \subseteq V$ a *condition* on subsets of V. We say that a set $T \subseteq V$ *satisfies* a condition (A^+, A^-) if $A^+ \subseteq T$ and $T \cap A^- = \emptyset$. A condition (A^+, A^-) is *trivial* if $A^+ \cup A^- = \emptyset$. Otherwise, the condition (A^+, A^-) is *non-trivial.*

Let $\mathcal{H}$ be a hypergraph and let A be a condition. By $\mathcal{H}_A$ we denote the hypergraph obtained from $\mathcal{H}$ by:

1. removing every edge E of $\mathcal{H}$ such that $E \cap A^+ \neq \emptyset$
2. removing from all the remaining edges of $\mathcal{H}$ elements that are in A^-
3. eliminating multiple edges.

Note that $\emptyset$ may be an edge in $\mathcal{H}_A$.

We have the following property of hypergraphs $\mathcal{H}$ and $\mathcal{H}_A$, which provides a basis for inductive arguments concerning properties of transversals.

Lemma 2.3 *Let $\mathcal{H}$ be a hypergraph and let A be a condition on subsets of $V(\mathcal{H})$. If $X \subseteq V(\mathcal{H})$ is a minimal transversal of $\mathcal{H}$ and X satisfies A, then $X \setminus A^+$ is a minimal transversal of $\mathcal{H}_A$.*

Proof: Let F be an edge of $\mathcal{H}_A$. Then, there is an edge E of $\mathcal{H}$ such that $E \cap A^+ = \emptyset$ and $F = E \setminus A^-$. Since X is a transversal of $\mathcal{H}$, $X \cap E \neq \emptyset$. Since X satisfies A, $X \cap A^- = \emptyset$. Thus, $X \cap F \neq \emptyset$. Finally, since $E \cap A^+ = \emptyset$, it follows that $F \cap A^+ = \emptyset$. Consequently, we have that $(X \setminus A^+) \cap F \neq \emptyset$. Since F is an arbitrary edge of $\mathcal{H}_A$, $X \setminus A^+$ is a transversal of $\mathcal{H}_A$.

Let $Y \subseteq X \setminus A^+$ be a transversal of $\mathcal{H}_A$. Since $X \cap A^- = \emptyset$ and $A^+ \cap A^- = \emptyset$, $Y \cup A^+$ satisfies A. It is now easy to check that $Y \cup A^+ \subseteq X$ and that $Y \cup A^+$ is a transversal of $\mathcal{H}$. By the minimality of X, $Y \cup A^+ = X$. Moreover, since $Y \cap A^+ = \emptyset$, $Y = X \setminus A^+$. It follows that $X \setminus A^+$ is a minimal transversal of $\mathcal{H}_A$. □

We note that it may happen that $\mathcal{H}_A$ contains the empty edge. In such case, $\mathcal{H}_A$ has no transversals and consequently, $\mathcal{H}$ has no transversals satisfying A. It may also happen that $\mathcal{H}_A$ is empty. If that is the case, A^+ is the *only* minimal transversal of $\mathcal{H}$ that satisfies A.

A key concept for the proof of Theorem 2.2 is that of a *complete* collection of conditions. Let $\mathcal{H}$ be a hypergraph. A *non-empty* family $\mathcal{A}$ of *non-trivial* conditions is *complete* for $\mathcal{H}$ if every minimal transversal of $\mathcal{H}$ satisfies at least one condition $A \in \mathcal{A}$. The family $\mathcal{A} = \{(\{a\}, \emptyset), (\emptyset, \{a\})\}$, where $a \in V(\mathcal{H})$, is an example of a complete family of conditions.

From now on we will fix attention on a class of hypergraphs $\mathcal{C}$ closed under the operations of removing edges and removing vertices from edges. In other words, we assume that if $\mathcal{H} \in \mathcal{C}$, and $\mathcal{H}'$ is obtained from $\mathcal{H}$ by removing from $\mathcal{H}$ some of its edges and by removing from some of the remaining edges some of their vertices, then $\mathcal{H}' \in \mathcal{C}$. We note that in particular the class $\mathcal{C}_{\leq 3}$ is closed under the operations of removing edges and removing vertices from edges.

We call a hypergraph $\mathcal{H}$ *proper* if $\mathcal{H} \neq \emptyset$ and if $\emptyset \notin \mathcal{H}$. A *descendant function for* $\mathcal{C}$ is a function assigning to each proper hypergraph in the class $\mathcal{C}$ a complete family of conditions. Let ρ be a descendant function. We use ρ to associate with each hypergraph $\mathcal{H} \in \mathcal{C}$ a labeled tree $\mathcal{T}^{\rho}_{\mathcal{H}}$ that helps to estimate the number of minimal transversals of $\mathcal{H}$. We will typically omit ρ from the notation, as ρ will always be clear from the context.

We define tree $\mathcal{T}_{\mathcal{H}}$ inductively. If $\emptyset \in \mathcal{H}$ or if $\mathcal{H} = \emptyset$, $\mathcal{T}_{\mathcal{H}}$ consists of a single node labeled with $\mathcal{H}$. Otherwise (that means, when $\mathcal{H}$ is proper), we form a new node, label it with $\mathcal{H}$ and make it the parent of all trees $\mathcal{T}_{\mathcal{H}_A}$, where $A \in \rho(\mathcal{H})$. Since for every $A \in \rho(\mathcal{H})$ the hypergraph $\mathcal{H}_A \in \mathcal{C}$ and $|V(\mathcal{H}_A)| \leq |V(\mathcal{H})|$, the definition is well founded. We denote the set of leaves of the tree $\mathcal{T}_{\mathcal{H}}$ by $L(\mathcal{T}_{\mathcal{H}})$.

Theorem 2.4 *Let ρ be a descendant functions for $\mathcal{C}$, where $\mathcal{C}$ is a class of hypergraphs closed under the operations of removing edges and removing vertices from edges. Then, for every hypergraph $\mathcal{H} \in \mathcal{C}$, the number of minimal transversals in a hypergraph $\mathcal{H}$ is at most $|L(\mathcal{T}^{\rho}_{\mathcal{H}})|$.*

Proof: We proceed by induction. If $\emptyset \in \mathcal{H}$ or if $\mathcal{H} = \emptyset$, the assertion is evident. Let us consider then a proper hypergraph $\mathcal{H}$ and let us assume that the assertion holds for every hypergraph $\mathcal{H}'$ with fewer vertices than $\mathcal{H}$.

Let X be a minimal transversal of $\mathcal{H}$. Since $\mathcal{H}$ is proper, $\rho(\mathcal{H})$ is well defined and is a complete family of conditions for $\mathcal{H}$. Thus, there is a set $A \in \rho(\mathcal{H})$ such that X satisfies A. By Lemma 2.3, $X = Y \cup A^+$, where Y is a minimal transversal for $\mathcal{H}_A$. It follows that the number of minimal transversals of $\mathcal{H}$ is at most $\sum_{A \in \rho(\mathcal{H})} t_A$, where t_A is the number of minimal transversals of $\mathcal{H}_A$. By the induction hypothesis, $t_A \leq |L(\mathcal{T}^{\rho}_{\mathcal{H}_A})|$, and the assertion follows. $\Box$

In the remainder of the paper, we will use Theorem 2.4 to prove Theorem 2.2. To obtain specific bound claimed in Theorem 2.2 we need a method to estimate the number of leaves in rooted trees. To this end, we adapt a method proposed in [4].

Let $\mathcal{T}$ be a rooted tree and let $L(\mathcal{T})$ be the set of leaves in $\mathcal{T}$. For a node x in $\mathcal{T}$, we denote by $D(x)$ the set of *directed* edges that link x with its children. For a leaf w of $\mathcal{T}$, we denote by $P(w)$ the set of *directed* edges on the unique path from the root of $\mathcal{T}$ to the leaf w. The following observation was shown in [4].

Proposition 2.5 [4] *Let p be a function assigning positive real numbers to edges of a rooted* *tree $\mathcal{T}$ such that for every internal node x in $\mathcal{T}$, $\sum_{e\in D(x)} p(e) = 1$. Then,*

$$|L(\mathcal{T})| \le \max_{w\in L(\mathcal{T})} (\prod_{e\in P(w)} p(e))^{-1}.$$

We will apply this result to derive an estimate on the number of leaves in the tree $\mathcal{T}_\mathcal{H}$. We define a *measure* to be any function μ that assigns to every hypergraph $\mathcal{H} \in \mathcal{C}$ a real number $\mu(\mathcal{H})$ such that $0 \le \mu(\mathcal{H}) \le |V(\mathcal{H})|$. Given a measure μ, a descendant function ρ (defined on $\mathcal{C}$) is *μ-compatible* if for every proper hypergraph $\mathcal{H}$ and for every $A \in \rho(\mathcal{H})$, $\mu(\mathcal{H}) - \mu(\mathcal{H}_A) > 0$. We denote the quantity $\mu(\mathcal{H}) - \mu(\mathcal{H}_A)$ by $\Delta(\mathcal{H}, \mathcal{H}_A)$.

Let μ be a measure and ρ a descendant function defined on $\mathcal{C}$. If ρ is μ-compatible then there is a unique positive real number $\tau \ge 1$ satisfying the equation

$$\sum_{A\in\rho(\mathcal{H})} \tau^{-\Delta(\mathcal{H},\mathcal{H}_A)} = 1. \tag{1}$$

Indeed, for $\tau \ge 1$ the left hand side of the equation (1) is a strictly decreasing continuous function of τ. Furthermore, its value for $\tau = 1$ is at least 1 (as $\rho(\mathcal{H}) \ne \emptyset$) and it approaches 0 when τ tends to infinity. We denote the number $\tau \ge 1$ satisfying (1) by $\tau_\mathcal{H}$ (to simplify notation, we omit references to ρ and μ; they will always be clear from the context).

We say that a descendant function ρ defined on $\mathcal{C}$ is *μ-bounded* by τ_0 if for every proper hypergraph $\mathcal{H} \in \mathcal{C}$, $\tau_\mathcal{H} \le \tau_0$.

Theorem 2.6 *Let μ be a measure and let ρ be a descendant function, both defined on a class $\mathcal{C}$ of hypergraphs such that $\mathcal{C}$ is closed under the operations of removing edges and removing vertices from edges. If ρ is μ-compatible and μ-bounded by τ_0 then for every hypergraph $\mathcal{H} \in \mathcal{C}$*

$$|L(\mathcal{T}_\mathcal{H})| \le \tau_0^{|V(\mathcal{H})|}. \tag{2}$$

Proof: Let $e = (x, y)$ be an edge in $\mathcal{T}_\mathcal{H}$. It follows from the definition of $\mathcal{T}_\mathcal{H}$ that there is a proper hypergraph $\mathcal{F}$ and a condition $A \in \rho(\mathcal{F})$ such that x and y are labeled with $\mathcal{F}$ and $\mathcal{F}_A$, respectively. We define $D(e) = \Delta(\mathcal{F}, \mathcal{F}_A)$. Since ρ is μ-compatible, $D(e) > 0$.

Let us now set $p(e) = \tau_\mathcal{F}^{-D(e)}$, where $\tau_\mathcal{F}$ is the root of the equation (1) for the hypergraph $\mathcal{F}$. Since, ρ is μ-bounded by τ_0, we have

$$p(e)^{-1} = \tau_\mathcal{F}^{D(e)} \le \tau_0^{D(e)}$$

(we recall that $\tau_\mathcal{F} \ge 1$).

Clearly, for every leaf $w \in \mathcal{T}_\mathcal{H}$ we have

$$\sum_{e\in P(w)} D(e) = \mu(\mathcal{H}) - \mu(\mathcal{W}) \le \mu(\mathcal{H}) \le |V(\mathcal{H})|,$$

where $\mathcal{W}$ is the hypergraph that labels w. Consequently,

$$(\prod_{e\in P(w)} p(e))^{-1} \le \prod_{e\in P(w)} \tau_0^{D(e)} = \tau_0^{\sum_{e\in P(w)} D(e)} \le \tau_0^{|V(\mathcal{H})|}.$$

Since the function p satisfies the assumptions of Proposition 2.5, the assertion follows. $\square$

Thus, to bound the number of leaves in the tree $\mathcal{T}_\mathcal{H}$ by $\tau_0^{|V(\mathcal{H})|}$, we need to define a measure μ and a descendant function ρ satisfying the assumptions of Theorem 2.6. For the class $\mathcal{C}_{\le 3}$, which is of the primary interest to us here, we have the following lemma.

Lemma 2.7 *There is a measure μ defined for every hypergraph in the class $\mathcal{C}_{\leq 3}$ and a descendant function ρ for $\mathcal{C}_{\leq 3}$ such that ρ is μ-compatible and μ-bounded by* 1.6701.. .

Proof: Let $\mathcal{H}$ be a hypergraph from the class $\mathcal{C}_{\leq 3}$. We denote by $k(\mathcal{H})$ the maximum number of pairwise disjoint 2-element edges in $\mathcal{H}$. We set

$$\mu(\mathcal{H}) = |V(\mathcal{H})| - \alpha k(\mathcal{H}),$$

where $\alpha = 0.1950..$ (we discuss in the appendix the basis for that choice of α). Clearly, μ is a measure on the class $\mathcal{C}_{\leq 3}$.

To define a descendant function ρ, we proceed in two steps.

I. We first define $\rho(\mathcal{F})$ for every *proper* hypergraph $\mathcal{F} \in \mathcal{C}_{\leq 3}$ such that no 3-element edge in $\mathcal{F}$ contains a 2-element edge of $\mathcal{F}$. Moreover, we do it so that $\Delta(\mathcal{F}, \mathcal{F}_A) > 0$, for every $A \in \rho(\mathcal{F})$, and $\tau_{\mathcal{F}} \leq 1.6701..$. (This is the tedious part of the argument. We provide the details for this step in the appendix.)

II. Next, we extend the definition from Step I to all proper hypergraphs in the class $\mathcal{C}_{\leq 3}$. Let $\mathcal{H}$ be a proper hypergraph in $\mathcal{C}_{\leq 3}$. We define $\mathcal{F}$ to be the hypergraph obtained by removing from $\mathcal{H}$ its every 3-element edge that contains a 2-element edge (if there are no such edges, $\mathcal{F} = \mathcal{H}$). Clearly, $\mathcal{F}$ is proper and satisfies the assumptions needed in Step I. We set $\rho(\mathcal{H}) = \rho(\mathcal{F})$.

Since $\mathcal{H}$ and $\mathcal{F}$ have the same transversals, $\rho(\mathcal{H})$ is indeed a complete collection of conditions for $\mathcal{H}$ and, consequently, ρ is a descendant function. Moreover, since $k(\mathcal{H}) = k(\mathcal{F})$ and $|V(\mathcal{H})| \geq |V(\mathcal{F})|$, for every $A \in \rho(\mathcal{H})$ $(= \rho(\mathcal{F}))$,

$$\Delta(\mathcal{H}, \mathcal{H}_A) \geq \Delta(\mathcal{F}, \mathcal{F}_A). \tag{3}$$

Since the construction in Step I ensures $\Delta(\mathcal{F}, \mathcal{F}_A) > 0$, ρ is μ-compatible. The equation (3) also implies that $\tau_{\mathcal{H}} \leq \tau_{\mathcal{F}}$. Thus, by properties of the construction from Step I, $\tau_{\mathcal{H}} \leq 1.6701..$. In other words, ρ is μ-bounded by 1.6701.. . □

Clearly, Theorem 2.2 (and so, also Theorem 2.1) follow directly from Theorem 2.6 and Lemma 2.7.

The construction and the argument needed for Step I in the proof of Lemma 2.7 consist of a tedious case analysis. We present them in the appendix.

Acknowledgments

This research was supported by the National Science Foundation under Grant No. 0097278.

References

[1] J. M. Griggs, C. M. Grinstead and D. R. Guichard. The number of maximal independent sets in a connected graph. *Discrete Mathematics*, 68(2-3):211-220, 1988.

[2] Z. Füredi. The number of maximal independent sets in connected graphs. *Journal of Graph Theory* 11(4):463-470, 1987.

[3] M. Hujter and Z. Tuza. The number of maximal independent sets in triangle-free graphs. *SIAM Journal on Discrete Mathematics*, 6(2):284-288, 1993.

[4] O. Kullmann. New methods for 3-SAT decision and worst-case analysis. Theoretical Computer Science, pages 1-72, 1999.

[5] V. Lifschitz. Circumscriptive theories: a logic-based framework for knowledge representation. *Journal of Philosophical Logic* 17(4):391-441, 1988.

[6] Z. Lonc and M. Truszczyński. Computing minimal models, stable models, and answer sets. *Lecture Notes in Computer Science* 2916:209-223, 2003.

[7] J. McCarthy. Circumscription - a form of non-monotonic reasoning. *Artificial Intelligence*, 13(1-2):27-39, 1980.

[8] J. W. Moon and L. Moser. On cliques in graphs. *Israel Journal of Mathematics*, 3:23-28, 1965.

[9] B. Sagan. A note on independent sets in trees. *SIAM Journal on Discrete Mathematics*, 1(1):105-108, 1988.

[10] I. Tomescu. Le nombre maximum de cliques et de recouvrements par cliques des hypergraphes chromatiques complets. *Discrete Mathematics* 37(2-3):263-277, 1981.

[11] H. S. Wilf. The number of maximal independent sets in a tree. *SIAM Journal on Algebraic and Discrete Methods* 7:125-130, 1986.

Appendix: Proof of Lemma 2.7

We complete here the proof of Lemma 2.7. Specifically, for every proper hypergraph $\mathcal{F}$ in $\mathcal{C}_{\leq 3}$ such that no 3-element edge of $\mathcal{F}$ contains a 2-element edge of $\mathcal{F}$, we define a complete collection of conditions $\rho(\mathcal{F})$ and show that:

1. $\Delta(\mathcal{F}, \mathcal{F}_A) > 0$, for every $A \in \rho(\mathcal{F})$, and
2. $\tau_{\mathcal{F}} \leq 1.6701..$.

The definition consists of several cases that reflect the structure of the hypergraph $\mathcal{F}$. In each case, we assume that none of the cases considered earlier applies and we provide an explicit complete collection $\rho(\mathcal{F})$ of conditions for $\mathcal{F}$. Then, for every $A \in \rho(\mathcal{F})$, we find a bound $k_{A,\mathcal{F}}$ such that $\Delta(\mathcal{F}, \mathcal{F}_A) \geq k_{A,\mathcal{F}}$. In each case, it will be clear that $k_{A,\mathcal{F}}$'s are positive. In each case, we also find a unique positive real number $\tau'_{\mathcal{F}}$ that satisfies the equation

$$\sum_{A \in \rho(\mathcal{F})} \tau^{-k_{A,\mathcal{F}}} = 1. \tag{4}$$

and show that $\tau'_{\mathcal{F}} \leq 1.6701..$. Since $\Delta(\mathcal{F}, \mathcal{F}_A) \geq k_{A,\mathcal{F}} > 0$, it follows that $\tau_{\mathcal{F}} \leq \tau'_{\mathcal{F}} \leq 1.6701..$ (we recall that $\tau_{\mathcal{F}}$ denotes the positive root of the equation (1)).

Case 1 covers hypergraphs $\mathcal{F}$ containing a 1-element edge. Case 2 covers hypergraphs $\mathcal{F}$, which contain 2-element edges and some two among them share a vertex. Cases 3 - 6 address the possibility, when $\mathcal{F}$ contains 2-element edges and they are all pairwise disjoint. That assumption makes estimating $k(\mathcal{F})$ easy. In the remaining cases, we assume that $\mathcal{F}$ consists of 3-element edges only. In these cases, for a vertex a of $\mathcal{F}$, by $\Gamma(a)$ we denote the graph induced by the edges bc such that abc is an edge in $\mathcal{F}$. In Case 7, we assume that for some vertex a, $\Gamma(a)$ has maximum degree at least 5. Case 8 covers hypergraphs with a vertex a such that $\Gamma(a)$ has maximum degree 4 or 3. Cases 9 - 15 cover situations when $\mathcal{F}$ contains a vertex a such that the maximum degree of $\Gamma(a)$ is 2 or 1. These cases do not cover hypergraphs $\mathcal{F}$, in which for every vertex a, $\Gamma(a)$ is isomorphic to one of three graphs: the graph whose components are a triangle and a single edge, the graph whose components are a 3-edge path and a single edge, and the graph whose components are three single edges. Case 16 covers all such hypergraphs $\mathcal{F}$.

Let us now explain the choice of a particular value of α in the definition of the measure $\mu(S) = |V(S)| - \alpha k(S)$. The goal is to choose α so that the maximum of the solutions of the equation (4) over all cases considered in the definition of ρ be as small as possible. It turns out that Cases 5(iii) and 9 are, in a sense, "extremal". In Case 5(iii) of the definition of $\rho(S)$, the equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-4+3\alpha} + 2\tau^{-6+4\alpha} + \tau^{-8+5\alpha} = 1. \tag{5}$$

In Case 9, the equation (4) becomes

$$\tau^{-1} + \tau^{-1-4\alpha} = 1. \tag{6}$$

The positive root $\tau_1(\alpha)$ of the equation (5) satisfies the inequality $\tau_1(\alpha) > 1$ and grows, when α grows from 0 to 1. On the other hand, the positive root $\tau_2(\alpha)$ of the equation (6) decreases, when α grows from 0 to 1 and satisfies the inequalities $1 < \tau_2(\alpha) \leq 2$. The larger of the roots $\tau_1(\alpha)$ and $\tau_2(\alpha)$ is minimized when $\tau = \tau_1(\alpha) = \tau_2(\alpha)$. Equation (6) implies

$$\tau^{\alpha} = (\tau - 1)^{-1/4}. \tag{7}$$

Substituting into (5) yields, after some simplification,

$$\tau^8 - \tau^7 + (\tau-1)^{1/2}\tau^4 + 2(\tau-1)^{1/4}\tau^2 + 1 - \tau^8(\tau-1)^{5/4} = 0.$$

Substituting $x = (\tau - 1)^{1/4}$, we get

$$(x^4+1)^8 - (x^4+1)^7 + x^2(x^4+1)^4 + 2x(x^4+1)^2 + 1 - (x^4+1)^8 x^5 = 0.$$

As $1 < \tau \leq 2$, $0 < x \leq 1$. The root $x \in (0,1]$ of the polynomial in the left-hand side of the equation above satisfies $0.90478 < x < 0.90479$. Thus, $\tau = 1 + x^4 = 1.6701.. \approx 1.6702$. By (7) we get $\alpha = \frac{\ln(\tau-1)}{-4\ln\tau} = 0.1950..$. It can be checked by direct computations that in all remaining cases, if $\alpha = 0.1950..$, then the roots of the equations (4) are smaller than 1.67.

In the remainder of the proof we write conditions as sets of expressions of the form a and $\bar{b}$, where a and b are vertices. That is, we identify a condition A with the set

$$A^+ \cup \{\bar{a} \colon a \in A^-\}.$$

For instance, the condition $(\{a, c\}, \{b\})$ can be denoted as $\{a, \bar{b}, c\}$ or $\{a, c, \bar{b}\}$ (the order in which we enumerate the elements is immaterial).

Case 1. There is a 1-element edge $\{a\}$ in $\mathcal{F}$.

Let $\mathcal{A} = \{\{a\}\}$. Since each transversal of $\mathcal{F}$ contains the vertex a, $\mathcal{A}$ is a complete family of conditions for $\mathcal{F}$. We define $\rho(\mathcal{F}) = \mathcal{A}$.

Let M be a set of $k(\mathcal{F})$ pairwise disjoint 2-element edges in $\mathcal{F}$ (we recall that $k(\mathcal{F})$ denotes the cardinality of a largest pairwise disjoint collection of 2-element edges in $\mathcal{F}$). Every edge of M that does not contain a is an edge of the hypergraph $\mathcal{F}_{\{a\}}$. Thus, $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$. Since $|V(\mathcal{F}_{\{a\}})| \leq |V(\mathcal{F})| - 1$,

$$\Delta(\mathcal{F}, \mathcal{F}_{\{a\}}) = \mu(\mathcal{F}) - \mu(\mathcal{F}_{\{a\}}) = |V(\mathcal{F})| - \alpha k(\mathcal{F}) - (|V(\mathcal{F}_{\{a\}})| - \alpha k(\mathcal{F}_{\{a\}})) \geq 1 - \alpha.$$

We set $k_{\{a\},\mathcal{F}} = 1 - \alpha$. Clearly $k_{\{a\},\mathcal{F}} > 0$.

The equation (4) specializes to

$$\tau^{-1+\alpha} = 1.$$

We have $\tau'_{\mathcal{F}} = 1$ and, consequently, $\tau_{\mathcal{F}} \leq 1$.

Comment. According to our discussion above, we will assume from now on that $\mathcal{F}$ has no 1-element edges.

Case 2. There is a pair of 2-element edges in $\mathcal{F}$, which have a common vertex.

Let M be a largest set of pairwise disjoint 2-element edges in $\mathcal{F}$. Because of the assumption we adopt for Case 2, there is a 2-element edge, say e_1 such that $e_1 \notin M$. Since M is a largest set of pairwise disjoint 2-element edges in $\mathcal{F}$, there is a 2-element edge, say e_2, in M such that e_1 and e_2 have a common vertex. Without loss of generality, $e_1 = ab$ and $e_2 = ac$, for some vertices a, b and c of $\mathcal{F}$.

Clearly, the family $\mathcal{A}' = \{\{a\}, \{\bar{a}\}\}$ is complete. Every transversal satisfying the condition $\{\bar{a}\}$ must contain the vertices b and c because it intersects the edges ab and ac. Thus, every such transversal satisfies the condition $\{\bar{a}, b, c\}$ and the family

$$\mathcal{A} = \{\{a\}, \{\bar{a}, b, c\}\}$$

is complete for $\mathcal{F}$, as well. We define $\rho(\mathcal{F}) = \mathcal{A}$.

The hypergraph $\mathcal{F}_{\{a\}}$ contains all 2-element edges of M except for ac. Thus, $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$. Moreover, every edge of M that does not contain a nor b is an edge of $\mathcal{F}_{\{\bar{a},b,c\}}$. Thus, $k(\mathcal{F}_{\{\bar{a},b,c\}}) \geq k(\mathcal{F}) - 2$. Since $|V(\mathcal{F}_{\{a\}})| \leq |V(\mathcal{F})| - 1$ and $|V(\mathcal{F}_{\{\bar{a},b,c\}})| \leq |V(\mathcal{F})| - 3$, we have

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 3 - 2\alpha & \text{if } A = \{\bar{a}, b, c\}. \end{cases}$$

We set $k_{A,\mathcal{F}}$ to be the bound appearing in the appropriate case of the definition above.

The equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-3+2\alpha} = 1.$$

Assuming $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.58$ and, consequently, $\tau_{\mathcal{F}} \leq 1.58$.

Comment. From now on we will assume that 2-element edges in $\mathcal{F}$ are vertex-disjoint. We will denote this set of edges by M.

In addition, when establishing lower bounds $k_{A,\mathcal{F}}$ on $\Delta(\mathcal{F}, \mathcal{F}_A)$ we will always use the inequality $|V(\mathcal{F}_A)| \leq |V(\mathcal{F})| - |A|$ and we will not state that fact explicitly anymore. We always specify conditions by enumerating their elements. In all cases, it is easy to show that all vertices involved in the specification of a condition are in fact different and we will typically omit these arguments. Thus, evaluating $|A|$ is straightforward.

Case 3. There is a vertex, say a, that belongs to a 2-element edge, say $e = ab$ in $\mathcal{F}$, and to no other edge of $\mathcal{F}$.

The family $\mathcal{A}' = \{\{a\}, \{\bar{a}\}\}$ is complete for $\mathcal{F}$. Every minimal transversal satisfying $\{a\}$ satisfies $\{a, \bar{b}\}$. Indeed, if a transversal T of $\mathcal{F}$ contains both a and b then $T - \{a\}$ is a transversal of $\mathcal{F}$, too (as e is the only edge in $\mathcal{F}$ containing a). Furthermore, every transversal of $\mathcal{F}$ satisfying $\{\bar{a}\}$ satisfies $\{\bar{a}, b\}$ because it must intersect e. Therefore, the family

$$\mathcal{A} = \{\{a, \bar{b}\}, \{\bar{a}, b\}\}$$

is complete for $\mathcal{F}$ and we set $\rho(\mathcal{F}) = \mathcal{A}$.

Clearly, all 2-element edges of $\mathcal{F}$ except for e are still 2-element edges in both $\mathcal{F}_{\{a,\bar{b}\}}$ and $\mathcal{F}_{\{\bar{a},b\}}$. Since by Case 2, all 2-element edges in $\mathcal{F}$ form an independent set, $k(\mathcal{F}_{\{a,\bar{b}\}}) \geq k(\mathcal{F}) - 1$ and $k(\mathcal{F}_{\{\bar{a},b\}}) \geq k(\mathcal{F}) - 1$. Hence, for $A \in \mathcal{A}$, we have

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq 2 - \alpha.$$

We set $k_{A,\mathcal{F}} = 2 - \alpha$, for every $A \in \mathcal{A}$, and note that $k_{A,\mathcal{F}} > 0$.

The equation (4) specializes to

$$2\tau^{-2+\alpha} = 1.$$

Assuming $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.47$ and, consequently, $\tau_{\mathcal{F}} \leq 1.47$.

Comment. From now on we will not explicitly state the numbers $k_{A,\mathcal{F}}$. We will specify them implicitly in inequalities bounding $\Delta(\mathcal{F}, \mathcal{F}_A)$ from below. In each case, it is straightforward to see that the numbers are positive, due to the fact that $\alpha < 1$.

Case 4. There is a vertex a in $\mathcal{F}$ that belongs to a 2-element edge, say $e_1 = ab$, and to exactly one 3-element edge, say $e_2 = acd$.

The collection $\mathcal{A}' = \{\{\bar{a}\}, \{a, \bar{b}\}, \{a, b\}\}$ is complete for $\mathcal{F}$. Every transversal of $\mathcal{F}$ satisfying $\{\bar{a}\}$ satisfies $\{\bar{a}, b\}$ (as it intersects e_1). Let T be a minimal transversal for $\mathcal{F}$ satisfying $\{a, b\}$. If T contains c or d then, since e_1 and e_2 are the only two edges in $\mathcal{F}$ that contain a, $T - \{a\}$ is a transversal for $\mathcal{F}$, contrary to the minimality of T. Thus, every minimal transversal for $\mathcal{F}$ satisfying $\{a, b\}$ satisfies $\{a, b, \bar{c}, \bar{d}\}$. It follows that

$$\mathcal{A} = \{\{\bar{a}, b\}, \{a, \bar{b}\}, \{a, b, \bar{c}, \bar{d}\}\}.$$

is complete and we define $\rho(\mathcal{F}) = \mathcal{A}$.

Clearly, all 2-element edges of $\mathcal{F}$ except for e_1 are 2-element edges in $\mathcal{F}_{\{\bar{a},b\}}$ and $\mathcal{F}_{\{a,\bar{b}\}}$. Thus, $k(\mathcal{F}_{\{\bar{a},b\}}) \geq k(\mathcal{F}) - 1$ and $k(\mathcal{F}_{\{a,\bar{b}\}}) \geq k(\mathcal{F}) - 1$. Moreover, every 2-element edge of $\mathcal{F}$ except for e_1 and those 2-element edges that contain c or d are 2-element edges in $\mathcal{F}_{\{a,b,\bar{c},\bar{d}\}}$. Consequently, $k(\mathcal{F}_{\{a,b,\bar{c},\bar{d}\}}) \geq k(\mathcal{F}) - 3$. Hence,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 2 - \alpha & \text{if } A = \{\bar{a}, b\}, \{a, \bar{b}\} \\ 4 - 3\alpha & \text{if } A = \{a, b, \bar{c}, \bar{d}\}. \end{cases}$$

The equation (4) specializes to

$$2\tau^{-2+\alpha} + \tau^{-4+3\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.65$ and, consequently, $\tau_{\mathcal{F}} \leq 1.65$.

Comment. We note that in all cases the arguments to show that $\mathcal{A}$ is a complete collection of conditions are similar. We typically start with a complete collection $\mathcal{A}'$ of conditions. Then, we extend (strengthen) some of the conditions in $\mathcal{A}'$. For instance, if A is a condition and there is an edge $e \in \mathcal{F}$ such that $e \setminus A^- = \{a\}$, for some vertex a, then a must belong to every transversal satisfying A. Consequently, A can be replaced with the condition $(A^+ \cup \{a\}, A^-)$. The second type of an argument to extend conditions that we use here exploits the property of minimality. That approach applies when we explicitly know all edges in $\mathcal{F}$ that contain a vertex a. If a transversal T satisfies a condition A such that $a \in A^+$, then the minimality of T implies that $T - \{a\}$ is not a transversal. Thus, it must not intersect at least one edge in $\mathcal{F}$ that contains a. That implies that some vertices must be included in A^-. We used this method in Cases 3 and 4. In the remainder of the proof, we will typically omit arguments of the first type and only sketch arguments of the second type.

Case 5. There is a vertex a in $\mathcal{F}$ that occurs in a 2-element edge, say $e_1 = ab$, and two 3-element edges, say $e_2 = acd$ and $e_3 = afg$, such that $c, d \neq f, g$.

Subcase (i). The vertices c, d, f, g do not belong to any 2-element edges.

Clearly, $\mathcal{A}' = \{\{a\}, \{\bar{a}\}\}$ is a complete family of conditions. Since $\{\bar{a}\}$ can be extended (by means of the first method) to $\{\bar{a}, b\}$, the family

$$\mathcal{A} = \{\{a\}, \{\bar{a}, b\}\}$$

is complete, as well, and we set $\rho(\mathcal{F}) = \mathcal{A}$.

All 2-element edges of $\mathcal{F}$ except for e_1 are 2-element edges in $\mathcal{F}_{\{a\}}$. Thus, $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$. Moreover, e_1 is not a 2-element edge of $\mathcal{F}_{\{\bar{a},b\}}$ while $e_4 = cd$ and $e_5 = fg$ are. Since no 2-element edge of $\mathcal{F}$ is a subset of a 3-element edge of $\mathcal{F}$, e_4 and e_5 are not edges of $\mathcal{F}$ and, by the assumptions of this subcase, do not have common vertices with 2-element edges of $M - \{e_1\}$. Hence, $k(\mathcal{F}_{\{\bar{a},b\}}) \geq k(\mathcal{F}) + 1$. It follows that

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 2 + \alpha & \text{if } A = \{\bar{a}, b\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-2-\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.66$ and, consequently, $\tau_{\mathcal{F}} \leq 1.66$.

Subcase (ii). The vertices c and d or the vertices f and g do not belong to any 2-element edge. Without loss of generality, we assume that the latter holds.

By Case 5(i), we can assume that at least one of c and d, say c belongs to a 2-element edge, say $e_4 = ch$. We note that $d \neq h$ (otherwise, e_4 would be a subset of e_2).

The family $\mathcal{A}' = \{\{a\}, \{\bar{a}, c\}, \{\bar{a}, \bar{c}\}\}$ is complete. The sets $\{\bar{a}, c\}$ and $\{\bar{a}, \bar{c}\}$ can be extended to $\{\bar{a}, c, b\}$ and $\{\bar{a}, \bar{c}, b, d, h\}$, respectively. Hence, the family

$$\mathcal{A} = \{\{a\}, \{\bar{a}, c, b\}, \{\bar{a}, \bar{c}, b, d, h\}\}$$

is complete and we set $\rho(\mathcal{F}) = \mathcal{A}$.

All 2-element edges of $\mathcal{F}$ except for e_1 are 2-element edges in $\mathcal{F}_{\{a\}}$. Thus, $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$. Moreover, all edges of M other than e_1 and e_4 are 2-element edges of $\mathcal{F}_{\{\bar{a},c,b\}}$. In addition, $e_5 = fg$ is also a 2-element edge of $\mathcal{F}_{\{\bar{a},c,b\}}$. Since e_5 is not an edge of $\mathcal{F}$ (as e_5 is a subset of e_3) and does not have common vertices with the 2-element edges of $M - \{e_1, e_4\}$, $k(\mathcal{F}_{\{\bar{a},c,b\}}) \geq k(\mathcal{F}) - 1$. Finally, all edges of $M - \{e_1, e_4\}$ that do not contain d are 2-element edges of $\mathcal{F}_{\{\bar{a},\bar{c},b,d,h\}}$. In addition, $e_5 = fg$ is also a 2-element edge of $\mathcal{F}_{\{\bar{a},\bar{c},b,d,h\}}$. Since e_5 is not an edge of $\mathcal{F}$ and does not have common vertices with the remaining 2-element edges of $\mathcal{F}_{\{\bar{a},\bar{c},b,d,h\}}$, $k(\mathcal{F}_{\{\bar{a},\bar{c},b,d,h\}}) \geq k(\mathcal{F}) - 2$. Hence,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 3 - \alpha & \text{if } A = \{\bar{a}, c, b\} \\ 5 - 2\alpha & \text{if } A = \{\bar{a}, \bar{c}, b, d, h\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-3+\alpha} + \tau^{-5+2\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Subcase (iii). At least one of the vertices c and d, say c, belongs to a 2-element edge in $\mathcal{F}$, say e_4, whose other vertex is not in e_3, and at least one of the vertices of f and g, say f, belongs to a 2-element edge, say e_5, whose other vertex is not in e_2.

Let $e_4 = ch$ and $e_5 = fj$. By our assumptions, h does not occur in e_3 and j does not occur in e_2, so a, b, c, d, f, g, h, j are pairwise different. Let

$$\mathcal{A} = \{\{a\}, \{\bar{a}, f, c, b\}, \{\bar{a}, \bar{f}, c, b, g, j\}, \{\bar{a}, f, \bar{c}, b, d, h\}, \{\bar{a}, \bar{f}, \bar{c}, b, d, g, h, j\}\}.$$

The family $\mathcal{A}' = \{\{a\}, \{\bar{a}, f, c\}, \{\bar{a}, \bar{f}, c\}, \{\bar{a}, f, \bar{c}\}, \{\bar{a}, \bar{f}, \bar{c}\}\}$ clearly is complete. Since $\mathcal{A}$ can be obtained from strengthening some of the conditions in $\mathcal{A}'$, $\mathcal{A}$ is complete, too, and we set $\rho(\mathcal{F}) = \mathcal{A}$.

All edges in $M - \{e_1\}$ are 2-element edges in $\mathcal{F}_{\{a\}}$ and so $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$. All edges in $M - \{e_1, e_4, e_5\}$ are 2-element edges in $\mathcal{F}_{\{\bar{a},f,c,b\}}$ and so $k(\mathcal{F}_{\{\bar{a},f,c,b\}}) \geq k(\mathcal{F}) - 3$. All edges in $M - \{e_1, e_4, e_5\}$ that do not contain g are 2-element edges in $\mathcal{F}_{\{\bar{a},\bar{f},c,b,g,j\}}$. Thus, $k(\mathcal{F}_{\{\bar{a},\bar{f},c,b,g,j\}}) \geq k(\mathcal{F}) - 4$. Similarly, all edges in $M - \{e_1, e_4, e_5\}$ that do not contain d are 2-element edges in $\mathcal{F}_{\{\bar{a},f\bar{c},b,d,h\}}$. Thus, $k(\mathcal{F}_{\{\bar{a},f,\bar{c},b,d,h\}}) \geq k(\mathcal{F}) - 4$. Finally, all edges in $M - \{e_1, e_4, e_5\}$ that do not contain d or g are 2-element edges in $\mathcal{F}_{\{\bar{a},\bar{f},\bar{c},b,d,g,h,j\}}$ and so $k(\mathcal{F}_{\{\bar{a},\bar{f},\bar{c},b,d,g,h,j\}}) \geq k(\mathcal{F}) - 5$. Hence,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 4 - 3\alpha & \text{if } A = \{\bar{a}, f, c, b\} \\ 6 - 4\alpha & \text{if } A = \{\bar{a}, \bar{f}, c, b, g, j\}, \{\bar{a}, f, \bar{c}, b, d, h\} \\ 8 - 5\alpha & \text{if } A = \{\bar{a}, \bar{f}, \bar{c}, b, d, g, h, j\}. \end{cases}$$

The equation (4) becomes

$$\tau^{-1+\alpha} + \tau^{-4+3\alpha} + 2\tau^{-6+4\alpha} + \tau^{-8+5\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} = 1.6701..$ and, consequently, $\tau_{\mathcal{F}} \leq 1.6701..$.

Subcase (iv). There is a 2-element edge e_4 with one vertex in e_2 and the other one in e_3, and both edges e_2 and e_3 have vertices which do not belong to any 2-element edges.

Without loss of generality, $e_4 = cf$. Clearly, d and g do not belong to any 2-element edges. Since $\mathcal{A}' = \{\{a\}, \{\bar{a}, c\}, \{\bar{a}, \bar{c}\}\}$ is complete for $\mathcal{F}$,

$$\mathcal{A} = \{\{a\}, \{\bar{a}, c, b\}, \{\bar{a}, \bar{c}, b, d, f\}\}$$

is complete for $\mathcal{F}$, as well. We define $\rho(\mathcal{F}) = \mathcal{A}$.

All edges in $M - \{e_1\}$ are 2-element edges in $\mathcal{F}_{\{a\}}$ and so $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$. All edges in $M - \{e_1, e_4\}$ are 2-element edges in $\mathcal{F}_{\{\bar{a},c,b\}}$. In addition, $e_5 = fg$ is also a 2-element edge in $\mathcal{F}_{\{\bar{a},c,b\}}$. Since $e_5 \notin \mathcal{F}$ and does not have common vertices with the 2-element edges of $M - \{e_1, e_4\}$, $k(\mathcal{F}_{\{\bar{a},c,b\}}) \geq k(\mathcal{F}) - 1$. Finally, all edges in $M - \{e_1, e_4\}$ are 2-element edges of $\mathcal{F}_{\{\bar{a},\bar{c},b,d,f\}}$ and so $k(\mathcal{F}_{\{\bar{a},\bar{c},b,d,f\}}) \geq k(\mathcal{F}) - 2$. Hence,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 3 - \alpha & \text{if } A = \{\bar{a}, c, b\} \\ 5 - 2\alpha & \text{if } A = \{\bar{a}, \bar{c}, b, d, f\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-3+\alpha} + \tau^{-5+2\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Subcase (v). There is a 2-element edge e_4 with one vertex in e_2 and the other vertex in e_3, and exactly one of the vertices of e_2 and e_3 does not belong to any 2-element edge.

Without loss of generality, $e_4 = cf$ and g is the vertex that does not belong to any 2-element edge in $\mathcal{F}$. Let the 2-element edge that contains d be $e_5 = dj$. Since the family $\mathcal{A}' = \{\{a\}, \{\bar{a}, f, d\}, \{\bar{a}, \bar{f}, d\}, \{\bar{a}, f, \bar{d}\}, \{\bar{a}, \bar{f}, \bar{d}\}\}$ is complete and the family

$$\mathcal{A} = \{\{a\}, \{\bar{a}, f, d, b\}, \{\bar{a}, \bar{f}, d, b, c, g\}, \{\bar{a}, f, \bar{d}, b, c, j\}, \{\bar{a}, \bar{f}, \bar{d}, b, c, g, j\}\}.$$

can be obtained by strengthening some of the conditions in $\mathcal{A}'$, $\mathcal{A}$ is complete, too. We set $\rho(\mathcal{F}) = \mathcal{A}$.

All edges in $M - \{e_1\}$ are 2-element edges in $\mathcal{F}_{\{a\}}$ and so $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$. Moreover, all edges in $M - \{e_1, e_4, e_5\}$ are 2-element edges in $\mathcal{F}_A$, for $A = \mathcal{F}_{\{\bar{a},f,d,b\}}$, $\mathcal{F}_{\{\bar{a},\bar{f},d,b,c,g\}}$, $\mathcal{F}_{\{\bar{a},f,\bar{d},b,c,j\}}$ and $\mathcal{F}_{\{\bar{a},\bar{f},\bar{d},b,c,g,j\}}$. Thus, $k(\mathcal{F}_A) \geq k(\mathcal{F}) - 3$ for all those sets A and, consequently,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 4 - 3\alpha & \text{if } A = \{\bar{a}, f, d, b\} \\ 6 - 3\alpha & \text{if } A = \{\bar{a}, \bar{f}, d, b, c, g\}, \{\bar{a}, f, \bar{d}, b, c, j\} \\ 7 - 3\alpha & \text{if } A = \{\bar{a}, \bar{f}, \bar{d}, b, c, g, j\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-4+3\alpha} + 2\tau^{-6+3\alpha} + \tau^{-7+3\alpha} = 1,$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Subcase (vi). There is a 2-element edge e_4 with one vertex in e_2 and the other one in e_3 and all vertices of e_2 and e_3 belong to 2-element edges.

Without the loss of generality, $e_4 = cf$. By Case 5(iii), we can assume also that the vertices d and g form a 2-element edge $e_5 = dg$.

Let us assume first that $e_1 = ab$, $e_2 = acd$ and $e_3 = afg$ are the only edges containing a. We observe that

$$\mathcal{A}' = \{\{\bar{a}\}, \{a, \bar{b}\}, \{a, b, \bar{c}, \bar{d}\}, \{a, b, \bar{f}, \bar{g}\}\}$$

is a complete collection of conditions. Indeed, let us suppose T is a minimal transversal in $\mathcal{F}$, which does not satisfy any of the conditions in $\mathcal{A}'$. Since T satisfies neither $\{\bar{a}\}$ nor $\{a, \bar{b}\}$, $a \in T$ and $b \in T$. Similarly, as T satisfies neither $\{a, b, \bar{c}, \bar{d}\}$ nor $\{a, b, \bar{f}, \bar{g}\}$, $c \in T$ or $d \in T$, and also $f \in T$ or $g \in T$. Since a belongs to the edges e_1, e_2 and e_3 only, $T - \{a\}$ is a transversal in $\mathcal{F}$, contrary to the minimality of T.

Since the conditions $\{\bar{a}\}$, $\{a, b, \bar{c}, \bar{d}\}$ and $\{a, b, \bar{f}, \bar{g}\}$ can be extended, it follows that

$$\mathcal{A} = \{\{\bar{a}, b\}, \{a, \bar{b}\}, \{a, b, \bar{c}, \bar{d}, f, g\}, \{a, b, \bar{f}, \bar{g}, c, d\}\}$$

is complete. We set $\rho(\mathcal{F}) = \mathcal{A}$.

All edges in $M - \{e_1\}$ are 2-element edges in $\mathcal{F}_{\{\bar{a},b\}}$ and $\mathcal{F}_{\{a,\bar{b}\}}$. Thus, $k(\mathcal{F}_{\{\bar{a},b\}}), k(\mathcal{F}_{\{a,\bar{b}\}}) \geq k(\mathcal{F}) - 1$. Moreover, all edges in $M - \{e_1, e_4, e_5\}$ are 2-element edges in $\mathcal{F}_{\{a,b,\bar{c},\bar{d},f,g\}}$ and $\mathcal{F}_{\{a,b,\bar{f},\bar{g},c,d\}}$. Thus, $k(\mathcal{F}_{\{a,b,\bar{c},\bar{d},f,g\}}), k(\mathcal{F}_{\{a,b,\bar{f},\bar{g},c,d\}}) \geq k(\mathcal{F}) - 3$ and, consequently,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 2 - \alpha & \text{if } A = \{\bar{a}, b\}, \{a, \bar{b}\} \\ 6 - 3\alpha & \text{if } A = \{a, b, \bar{c}, \bar{d}, f, g\}, \{a, b, \bar{f}, \bar{g}, c, d\}. \end{cases}$$

The equation (4) specializes to

$$2\tau^{-2+\alpha} + 2\tau^{-6+3\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.61$ and, consequently, $\tau_{\mathcal{F}} \leq 1.61$.

Suppose now that a belongs to some 3-element edge $e \neq e_2, e_3$. If e and e_2 have exactly one vertex in common (it must be a) then replacing e_3 by e we get Case 5(ii) or 5(iii).

Let e and e_2 have two vertices in common (say a and c). Replacing e_3 by e we get Case 6(i), which we consider below (and which is independent of Case 5).

Case 6. There is a vertex, say a, that occurs in a 2-element edge, say $e_1 = ab$ and two 3-element edges, say $e_2 = acd$ and $e_3 = afg$, such that e_2 and e_3 have 2 vertices in common.

Clearly, a is one of the vertices common to e_2 and e_3. Without loss of generality, $c = f$. Since $e_2 \neq e_3$, $d \neq g$.

Subcase (i). The vertex c belongs to a 2-element edge, say $e_4 = ch$.

We define

$$\mathcal{A} = \{\{a\}, \{\bar{a}, c, b\}, \{\bar{a}, \bar{c}, b, d, g, h\}\}.$$

Since $\mathcal{A}' = \{\{a\}, \{\bar{a}, c\}, \{\bar{a}, \bar{c}\}\}$ is complete and $\mathcal{A}$ can be obtained from $\mathcal{A}'$ by extending $\{\bar{a}, c\}$ and $\{\bar{a}, \bar{c}\}$, it follows that $\mathcal{A}$ is complete and we set $\rho(A) = \mathcal{A}$.

All edges in $M - \{e_1\}$ are 2-element edges in $\mathcal{F}_{\{a\}}$ and so $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$. All edges in $M - \{e_1, e_4\}$ are 2-element edges in $\mathcal{F}_{\{\bar{a},c,b\}}$. Thus, $k(\mathcal{F}_{\{\bar{a},c,b\}}) \geq k(\mathcal{F}) - 2$. Finally, all edges in $M - \{e_1, e_4\}$ that do not contain d or g are 2-element edges in $\mathcal{F}_{\{\bar{a},\bar{c},b,d,g,h\}}$ and so $k(\mathcal{F}_{\{\bar{a},\bar{c},b,d,g,h\}}) \geq k(\mathcal{F}) - 4$. Hence,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 3 - 2\alpha & \text{if } A = \{\bar{a}, c, b\} \\ 6 - 4\alpha & \text{if } A = \{\bar{a}, \bar{c}, b, d, g, h\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-3+2\alpha} + \tau^{-6+4\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Comment. As we noted, the argument for Case 6(i) concludes also the argument for Case 5.

Subcase (ii). The vertex c does not belong to any 2-element edge and at most one of the vertices d, g belongs to a 2-element edge.

We define

$$\mathcal{A} = \{\{a\}, \{\bar{a}, c, b\}, \{\bar{a}, \bar{c}, b, d, g\}\}.$$

As before, it is easy to see that $\mathcal{A}$ is complete and we set $\rho(\mathcal{F}) = \mathcal{A}$.

All edges in $M - \{e_1\}$ are 2-element edges in $\mathcal{F}_{\{a\}}$ and $\mathcal{F}_{\{\bar{a},c,b\}}$. Consequently, $k(\mathcal{F}_{\{a\}}) \geq k(\mathcal{F}) - 1$ and $k(\mathcal{F}_{\{\bar{a},c,b\}}) \geq k(\mathcal{F}) - 1$. Moreover, all edges in $M - \{e_1\}$ that do not contain d or g (by the assumption of the subcase, at most one edge is excluded by that condition), are 2-element edges in $\mathcal{F}_{\{\bar{a},\bar{c},b,d,g\}}$ and so $k(\mathcal{F}_{\{\bar{a},\bar{c},b,d,g\}}) \geq k(\mathcal{F}) - 2$. Hence,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 3 - \alpha & \text{if } A = \{\bar{a}, c, b\} \\ 5 - 2\alpha & \text{if } A = \{\bar{a}, \bar{c}, b, d, g\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-3+\alpha} + \tau^{-5+2\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Subcase (iii). The vertex c does not belong to any 2-element edge and both vertices d and g belong to (not necessarily different) 2-element edges.

First, we assume that e_1, e_2 and e_3 are the only edges that contain a. Clearly, the collection

$$\mathcal{A}' = \{\{a, \bar{b}\}, \{a, b\}, \{\bar{a}, c\}, \{\bar{a}, \bar{c}\}\}$$

is complete. Every minimal transversal T of $\mathcal{F}$ satisfying $\{a, b\}$ satisfies $\{a, b, \bar{c}\}$ (otherwise, $b, c \in T - \{a\}$, and $T - \{a\}$ is a transversal in $\mathcal{F}$, a contradiction). The conditions $\{\bar{a}, c\}$ and $\{\bar{a}, \bar{c}\}$ can also be extended and it follows that

$$\mathcal{A} = \{\{a, \bar{b}\}, \{a, b, \bar{c}\}, \{\bar{a}, c, b\}, \{\bar{a}, \bar{c}, b, d, g\}\}$$

is complete. We set $\rho(\mathcal{F}) = \mathcal{A}$.

All edges in $M - \{e_1\}$ are 2-element edges in $\mathcal{F}_{\{a,\bar{b}\}}$, $\mathcal{F}_{\{a,b,\bar{c}\}}$ and $\mathcal{F}_{\{\bar{a},c,b\}}$. Consequently, $k(\mathcal{F}_{\{a,\bar{b}\}})$, $k(\mathcal{F}_{\{a,b,\bar{c}\}})$, $k(\mathcal{F}_{\{\bar{a},c,b\}}) \geq k(\mathcal{F}) - 1$. Moreover, all edges in $M - \{e_1\}$ that do not contain d or g are 2-element edges in $\mathcal{F}_{\{\bar{a},\bar{c},b,d,g\}}$, so $k(\mathcal{F}_{\{\bar{a},\bar{c},b,d,g\}}) \geq k(\mathcal{F}) - 3$. Hence,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 2 - \alpha & \text{if } A = \{a, \bar{b}\} \\ 3 - \alpha & \text{if } A = \{a, b, \bar{c}\}, \{\bar{a}, c, b\} \\ 5 - 3\alpha & \text{if } A = \{\bar{a}, \bar{c}, b, d, g\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-2+\alpha} + 2\tau^{-3+\alpha} + \tau^{-5+3\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.66$ and, consequently, $\tau_{\mathcal{F}} \leq 1.66$.

Let us suppose now that e_1, e_2 and e_3 are not the only edges that contain a. Then a belongs to some 3-element edge $e \neq e_2, e_3$ (since a belongs to a 2-element edge e_1 it does not belong to any other 2-element edge in $\mathcal{F}$).

If a is the only common vertex of e and e_2, then Case 5 (completed when we completed Case 6(i)) applies. Thus, we can assume that e has two common vertices with e_2. If e contains d, then Case 6(i) applies. Thus, c is a common vertex of e and e_2. In the same way we argue that c is a common vertex of e and e_3. Thus, $e = ach$ (and, of course, $h \neq d, g$). Since, clearly, $\mathcal{A}' = \{\{a\}, \{\bar{a}, c\}, \{\bar{a}, \bar{c}\}\}$ is complete, it follows that

$$\mathcal{A} = \{\{a\}, \{\bar{a}, c, b\}, \{\bar{a}, \bar{c}, b, d, g, h\}\}$$

is complete, too. We set $\rho(\mathcal{F}) = \mathcal{A}$.

All edges in $M - \{e_1\}$ are 2-element edges in $\mathcal{F}_{\{a\}}$ and $\mathcal{F}_{\{\bar{a},c,b\}}$. Thus, $k(\mathcal{F}_{\{a\}})$, $k(\mathcal{F}_{\{\bar{a},c,b\}}) \geq k(\mathcal{F}) - 1$. Moreover, all edges in $M - \{e_1\}$ that do not contain d, g or h are 2-element edges in $\mathcal{F}_{\{\bar{a},\bar{c},b,d,g,h\}}$ and so $k(\mathcal{F}_{\{\bar{a},\bar{c},b,d,g,h\}}) \geq k(\mathcal{F}) - 4$. Hence,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 - \alpha & \text{if } A = \{a\} \\ 3 - \alpha & \text{if } A = \{\bar{a}, c, b\} \\ 6 - 4\alpha & \text{if } A = \{\bar{a}, \bar{c}, b, d, g, h\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1+\alpha} + \tau^{-3+\alpha} + \tau^{-6+4\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.64$ and, consequently, $\tau_{\mathcal{F}} \leq 1.64$.

Comment. From now on we will assume that $\mathcal{F}$ contains 3-element edges only, that is, from now on, $k(\mathcal{F}) = 0$.
In addition, for a vertex a, we denote by $\Gamma(a)$ the undirected graph induced by the set of edges $\{bc: \ abc \in \mathcal{F}\}$.
Case 7. There is a vertex $a \in \mathcal{F}$ such that $\Gamma(a)$ has a vertex of degree at least 5.

Let b be a vertex of degree at least 5 in $\Gamma(a)$ and let b_1, b_2, b_3, b_4, b_5 be neighbors of b in $\Gamma(a)$. We define $\mathcal{A}' = \{\{a\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}\}\}$. Clearly, $\mathcal{A}'$ is complete. Every transversal satisfying $\{\bar{a}, \bar{b}\}$ satisfies $\{\bar{a}, \bar{b}, b_1, b_2, b_3, b_4, b_5\}$ as it intersects the edges abb_i, $i = 1, 2, 3, 4, 5$. Hence, the family

$$\mathcal{A} = \{\{a\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}, b_1, b_2, b_3, b_4, b_5\}\}$$

is complete and we set $\rho(\mathcal{F}) = \mathcal{A}$.

It follows that (we recall that from Case 7 on we can assume that $k(\mathcal{F}) = 0$)

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2 & \text{if } A = \{\bar{a}, b\} \\ 7 & \text{if } A = \{\bar{a}, \bar{b}, b_1, b_2, b_3, b_4, b_5\}. \end{cases}$$

The equation (4) becomes

$$\tau^{-1} + \tau^{-2} + \tau^{-7} = 1.$$

We have $\tau'_{\mathcal{F}} \leq 1.66$ and, consequently, $\tau_{\mathcal{F}} \leq 1.66$.
Case 8. The maximum degree of a vertex in $\Gamma(a)$ is 4 or 3.

Let b be a vertex of maximum degree in $\Gamma(a)$. If the degree of b is 4 then we denote its neighbors by b_1, b_2, b_3 and b_4. If the degree of b is 3, we denote its neighbors by b_1, b_2 and b_3.
Subcase (i). $\Gamma(a)$ has a vertex of degree 4 and there are at least 5 edges in $\Gamma(a)$.

Let cd be an edge in $\Gamma(a)$ that is not incident to b. We define $\mathcal{A}' = \{\{a\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}\}\}$ and argue similarly as before that

$$\mathcal{A} = \{\{a\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}, b_1, b_2, b_3, b_4\}\}$$

is complete. We set $\rho(\mathcal{F}) = \mathcal{A}$.

The hypergraph $\mathcal{F}_{\{\bar{a}, b\}}$ contains the 2-element edge cd, so $k(\mathcal{F}_{\{\bar{a}, b\}}) \geq 1$. Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2 + \alpha & \text{if } A = \{\bar{a}, b\} \\ 6 & \text{if } A = \{\bar{a}, \bar{b}, b_1, b_2, b_3, b_4\}. \end{cases}$$

The equation (4) becomes

$$\tau^{-1} + \tau^{-2-\alpha} + \tau^{-6} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.64$ and, consequently, $\tau_{\mathcal{F}} \leq 1.64$.
Subcase (ii). $\Gamma(a)$ is a star.

Clearly, the family $\mathcal{A}' = \{\{a\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}\}\}$ is complete. The condition $\{\bar{a}, \bar{b}\}$ extends to $\{\bar{a}, \bar{b}, b_1, b_2, b_3\}$. Moreover, every minimal transversal T satisfying $\{a\}$ satisfies $\{a, \bar{b}\}$ (if not, $T - \{a\}$ is a transversal contrary to the minimality of T). Hence,

$$\mathcal{A} = \{\{a, \bar{b}\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}, b_1, b_2, b_3\}\}$$

is complete and we set $\rho(\mathcal{F}) = \mathcal{A}$.

We now have

$$\Delta(\mathcal{F},\mathcal{F}_A) \geq \begin{cases} 2 & \text{if } A = \{a,\bar{b}\},\{\bar{a},b\} \\ 5 & \text{if } A = \{\bar{a},\bar{b},b_1,b_2,b_3\}. \end{cases}$$

The equation (4) becomes

$$2\tau^{-2} + \tau^{-5} = 1.$$

We have $\tau'_{\mathcal{F}} \leq 1.52$ and, consequently, $\tau_{\mathcal{F}} \leq 1.52$.

Subcase (iii). The maximum degree of a vertex in $\Gamma(a)$ is 3, and for some vertex of maximum degree in $\Gamma(a)$, say b, $\Gamma(a)$ contains at least 2 independent edges not incident to b.

Let c_1d_1 and c_2d_2 be two independent edges in $\Gamma(a)$ that are not incident to b. The family $\mathcal{A}' = \{\{a\},\{\bar{a},b\},\{\bar{a},\bar{b}\}\}$ is complete for $\mathcal{F}$. Thus,

$$\mathcal{A} = \{\{a\},\{\bar{a},b\},\{\bar{a},\bar{b},b_1,b_2,b_3\}\}$$

is complete, too, and we set $\rho(\mathcal{F}) = \mathcal{A}$.

The hypergraph $\mathcal{F}_{\{\bar{a},b\}}$ contains the 2-element edges c_1d_1 and c_2d_2 and so $k(\mathcal{F}_{\{\bar{a},b\}}) \geq 2$. Thus,

$$\Delta(\mathcal{F},\mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2+2\alpha & \text{if } A = \{\bar{a},b\} \\ 5 & \text{if } A = \{\bar{a},\bar{b},b_1,b_2,b_3\}. \end{cases}$$

The equation (4) becomes

$$\tau^{-1} + \tau^{-2-2\alpha} + \tau^{-5} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.65$ and, consequently, $\tau_{\mathcal{F}} \leq 1.65$.

Subcase (iv). The maximum degree of a vertex in $\Gamma(a)$ is 3 and for some vertex of maximum degree in $\Gamma(a)$, say b, $\Gamma(a)$ contains at least 2 edges not incident to b.

Let c_1d' and c_2d'' be two edges in $\Gamma(a)$ that are not incident to b. By Case 8(iii), we can assume that $d' = d'' = d$ for some vertex d. Clearly, the collection $\mathcal{A}' = \{\{a\},\{\bar{a},b,d\},\{\bar{a},b,\bar{d}\},\{\bar{a},\bar{b}\}\}$ is complete. It follows that

$$\mathcal{A} = \{\{a\},\{\bar{a},b,d\},\{\bar{a},b,\bar{d},c_1,c_2\},\{\bar{a},\bar{b},b_1,b_2,b_3\}\}.$$

is complete, as well. We set $\rho(\mathcal{F}) = \mathcal{A}$.

We have

$$\Delta(\mathcal{F},\mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 3 & \text{if } A = \{\bar{a},b,d\} \\ 5 & \text{if } A = \{\bar{a},b,\bar{d},c_1,c_2\},\{\bar{a},\bar{b},b_1,b_2,b_3\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1} + \tau^{-3} + 2\tau^{-5} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.65$ and, consequently, $\tau_{\mathcal{F}} \leq 1.65$.

Subcase (v). The maximum degree of a vertex in $\Gamma(a)$ is 3 and for some vertex of maximum degree in $\Gamma(a)$, say b, $\Gamma(a)$ has exactly one edge not incident to b.

Let cd be the edge in $\Gamma(a)$ that is not incident to b. Clearly, $\mathcal{A}' = \{\{a,\bar{b}\},\{a,b\},\{\bar{a},b\},\{\bar{a},\bar{b}\}\}$ is complete. Thus,

$$\mathcal{A} = \{\{a,\bar{b}\},\{a,b,\bar{c},\bar{d}\},\{\bar{a},b\},\{\bar{a},\bar{b},b_1,b_2,b_3\}\}$$

is complete (for instance, we have that every minimal transversal T satisfying $\{a, b\}$ satisfies $\{a, b, \bar{c}, \bar{d}\}$ as otherwise, $T - \{a\}$ would be a transversal, contrary to the minimality of T). We set $\rho(\mathcal{F}) = \mathcal{A}$.

The hypergraph $\mathcal{F}_{\{\bar{a},b\}}$ contains the 2-element edge cd, so $k(\mathcal{F}_{\{\bar{a},b\}}) \geq 1$. Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 2 & \text{if } A = \{a, \bar{b}\} \\ 4 & \text{if } A = \{a, b, \bar{c}, \bar{d}\} \\ 2 + \alpha & \text{if } A = \{\bar{a}, b\} \\ 5 & \text{if } A = \{\bar{a}, \bar{b}, b_1, b_2, b_3\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-2} + \tau^{-4} + \tau^{-2-\alpha} + \tau^{-5} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.6$ and, consequently, $\tau_{\mathcal{F}} \leq 1.6$.

Comment. From now on we can assume that for every vertex a, $\Gamma(a)$ has maximum degree 1 or 2 (we do not need that assumption in Case 9 but all of the remaining cases in our argument do require it).

Case 9. $\Gamma(a)$ contains 4 independent edges.

Let c_1d_1, c_2d_2, c_3d_3, c_4d_4 be independent edges in $\Gamma(a)$. We define

$$\mathcal{A} = \{\{a\}, \{\bar{a}\}\}.$$

Clearly, $\mathcal{A}$ is complete and we set $\rho(\mathcal{F}) = \mathcal{A}$.

The hypergraph $\mathcal{F}_{\{\bar{a}\}}$ contains four independent 2-element edges c_id_i, $i = 1, 2, 3, 4$, so $k(\mathcal{F}_{\{\bar{a}\}}) \geq 4$. Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 1 + 4\alpha & \text{if } A = \{\bar{a}\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1} + \tau^{-1-4\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} = 1.6701..$ and, consequently, $\tau_{\mathcal{F}} \leq 1.6701..$. We note that Case 9 is one of the two extremal cases that we used to determine the optimal value for α.

Case 10. The maximum degree of a vertex in $\Gamma(a)$ is 2 and $\Gamma(a)$ has at least 5 edges.

Subcase (i). There is a pair of nonadjacent vertices of degree 2 in $\Gamma(a)$.

Let b and c be two nonadjacent vertices of degree 2 in $\Gamma(a)$. We denote by b_1 and b_2 the neighbors of b and by c_1 and c_2 the neighbors of c. We define

$$\mathcal{A} = \{\{a\}, \{\bar{a}, \bar{b}, b_1, b_2\}, \{\bar{a}, b, c\}, \{\bar{a}, b, \bar{c}, c_1, c_2\}\}$$

Starting with $\mathcal{A}' = \{\{a\}, \{\bar{a}, \bar{b}\}, \{\bar{a}, b, c\}, \{\bar{a}, b, \bar{c}\}\}$ and reasoning in a standard way, one can show that $\mathcal{A}$ is complete. We set $\rho(\mathcal{F}) = \mathcal{A}$.

Since maximum degree of a vertex in $\Gamma(a)$ is 2, there are in $\Gamma(a)$ at most two edges other that bb_1 and bb_2 which are incident to a vertex in $\{b, b_1, b_2\}$. Since $\Gamma(a)$ has at least 5 edges, there is an edge, say df, in $\Gamma(a)$ whose endvertices are different from b, b_1 and b_2. Thus, the hypergraph $\mathcal{F}_{\{\bar{a},\bar{b},b_1,b_2\}}$ contains the 2-element edge df and $k(\mathcal{F}_{\{\bar{a},\bar{b},b_1,b_2\}}) \geq 1$. Similarly, there is an edge, say gh, in $\Gamma(a)$ whose endvertices are different from b and c. Hence, the hypergraph $\mathcal{F}_{\{\bar{a},b,c\}}$ contains the 2-element edge gh, so $k(\mathcal{F}_{\{\bar{a},b,c\}}) \geq 1$. Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 4+\alpha & \text{if } A = \{\bar{a}, \bar{b}, b_1, b_2\} \\ 3+\alpha & \text{if } A = \{\bar{a}, b, c\} \\ 5 & \text{if } A = \{\bar{a}, b, \bar{c}, c_1, c_2\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1} + \tau^{-4-\alpha} + \tau^{-3-\alpha} + \tau^{-5} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.66$ and, consequently, $\tau_{\mathcal{F}} \leq 1.66$.

Subcase (ii). Vertices of degree 2 are pairwise adjacent in $\Gamma(a)$.

If there are no vertices of degree 2 in $\Gamma(a)$ then $\Gamma(a)$ contains 4 independent edges and Case 9 applies. Let b be a vertex of degree 2 in $\Gamma(a)$ and let b_1 and b_2 be the neighbors of b. We define

$$\mathcal{A} = \{\{a\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}, b_1, b_2\}\}.$$

The family $\mathcal{A}$ is, clearly, complete. We set $\rho(\mathcal{F}) = \mathcal{A}$.

Let us assume that some two edges in $\Gamma(a)$ that are not incident to b have a common vertex, say c. Then, the degree of c in $\Gamma(a)$ is 2 and none of the edges incident to c is incident to b. This contradicts the assumption of this subcase. Thus, the hypergraph $\mathcal{F}_{\{\bar{a},b\}}$ contains three pairwise disjoint 2-element edges and so $k(\mathcal{F}_{\{\bar{a},b\}}) \geq 3$. We have

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2+3\alpha & \text{if } A = \{\bar{a}, b\} \\ 4 & \text{if } A = \{\bar{a}, \bar{b}, b_1, b_2\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1} + \tau^{-2-3\alpha} + \tau^{-4} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Case 11. The maximum degree of a vertex in $\Gamma(a)$ is 2, $\Gamma(a)$ has 4 edges and there is a pair of nonadjacent vertices of degree 2 in $\Gamma(a)$.

Let b and c be two nonadjacent vertices of degree 2 in $\Gamma(a)$. We denote by b_1 and b_2 the neighbors of b, and by c_1 and c_2 the neighbors of c. Clearly, the collection of conditions

$$\mathcal{A}' = \{\{a, \bar{b}\}, \{a, b\}, \{\bar{a}, \bar{b}\}, \{\bar{a}, b, c\}, \{\bar{a}, b, \bar{c}\}\}$$

is complete. A standard reasoning shows that

$$\mathcal{A} = \{\{a, \bar{b}\}, \{a, b, \bar{c}\}, \{\bar{a}, \bar{b}, b_1, b_2\}, \{\bar{a}, b, c\}, \{\bar{a}, b, \bar{c}, c_1, c_2\}\}$$

is complete too (for instance, every minimal transversal T satisfying $\{a, b\}$ satisfies $\{a, b, \bar{c}\}$ for otherwise $T - \{a\}$ would be a transversal in $\mathcal{F}$, contrary to the minimality of T). We also note that since b and c are nonadjacent in $\Gamma(a)$, b, c, c_1, c_2 are pairwise different. Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 2 & \text{if } A = \{a, \bar{b}\} \\ 3 & \text{if } A = \{a, b, \bar{c}\}, \{\bar{a}, b, c\} \\ 4 & \text{if } A = \{\bar{a}, \bar{b}, b_1, b_2\} \\ 5 & \text{if } A = \{\bar{a}, \bar{c}, b, c_1, c_2\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-2} + 2\tau^{-3} + \tau^{-4} + \tau^{-5} = 1.$$

We have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Case 12. The maximum degree of a vertex in $\Gamma(a)$ is 2, $\Gamma(a)$ has 4 edges and there is exactly one vertex of degree 2 in $\Gamma(a)$.

Let b be the vertex of degree 2 in $\Gamma(a)$, let c and d be the neighbors of b in $\Gamma(a)$ and let f_1f_2, g_1g_2 be the two isolated edges in $\Gamma(a)$. We define

$$\mathcal{A} = \{\{a\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}, c, d\}\}.$$

Clearly, the family $\mathcal{A}$ is complete and we set $\rho(\mathcal{F}) = \mathcal{A}$.

Both hypergraphs $\mathcal{F}_{\{\bar{a},b\}}$ and $\mathcal{F}_{\{\bar{a},\bar{b},c,d\}}$ contain two independent 2-element edges f_1f_2 and g_1g_2. Thus, $k(\mathcal{F}_{\{\bar{a},b\}})$, $k(\mathcal{F}_{\{\bar{a},\bar{b},c,d\}}) \geq 2$ and we have

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a\} \\ 2 + 2\alpha & \text{if } A = \{\bar{a}, b\} \\ 4 + 2\alpha & \text{if } A = \{\bar{a}, \bar{b}, c, d\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1} + \tau^{-2-2\alpha} + \tau^{-4-2\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Case 13. The maximum degree of a vertex in $\Gamma(a)$ is 2 and $\Gamma(a)$ has 3 edges.

Let b be a vertex of degree 2 in $\Gamma(a)$. We denote by b_1 and b_2 the neighbors of b. Let cc_1 be the edge in $\Gamma(a)$ which is not incident to b. Clearly, the collection $\mathcal{A}' = \{\{a, \bar{b}\}, \{a, b\}, \{\bar{a}, \bar{b}\}, \{\bar{a}, b, c\}, \{\bar{a}, b, \bar{c}\}\}$ is complete. We define

$$\mathcal{A} = \{\{a, \bar{b}\}, \{a, b, \bar{c}, \bar{c_1}\}, \{\bar{a}, \bar{b}, b_1, b_2\}, \{\bar{a}, b, c\}, \{\bar{a}, b, \bar{c}, c_1\}\}$$

As in other cases, a standard reasoning shows that $\mathcal{A}$ is complete (for instance, every minimal transversal T satisfying $\{a, b\}$ satisfies $\{a, b, \bar{c}, \bar{c_1}\}$ for otherwise $T - \{a\}$ would be a transversal, contrary to the minimality of T). We set $\rho(\mathcal{F}) = \mathcal{A}$. We also note that as the edge cc_1 is not incident to b, the vertices b, c and c_1 are pairwise different. Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 2 & \text{if } A = \{a, \bar{b}\} \\ 3 & \text{if } A = \{\bar{a}, b, c\} \\ 4 & \text{if } A = \{a, b, \bar{c}, \bar{c_1}\}, \{\bar{a}, \bar{b}, b_1, b_2\}, \{\bar{a}, b, \bar{c}, c_1\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-2} + \tau^{-3} + 3\tau^{-4} = 1.$$

We have $\tau'_{\mathcal{F}} \leq 1.65$ and, consequently, $\tau_{\mathcal{F}} \leq 1.65$.

Case 14. The graph $\Gamma(a)$ has two edges and they are independent.

Let b_1b_2 and c_1c_2 be the two edges in $\Gamma(a)$. We define

$$\mathcal{A} = \{\{a, \bar{b_1}, \bar{b_2}\}, \{a, \bar{c_1}, \bar{c_2}\}, \{\bar{a}\}\}.$$

Let T be a minimal transversal in $\mathcal{F}$. If $a \notin T$, then T satisfies $\{\bar{a}\}$. Thus, let $a \in T$ and let us assume that T does not satisfy $\{a, \bar{b_1}, \bar{b_2}\}$. Then, $T - \{a\}$ intersects the edge ab_1b_2. Since ab_1b_2 and ac_1c_2 are the only two edges in $\mathcal{F}$ that contain a and since $T - \{a\}$ is not a transversal, it follows that $T - \{a\}$ does not intersect ac_1c_2. Thus, T satisfies $\{a, \bar{c_1}, \bar{c_2}\}$. It follows that $\mathcal{A}$ is complete and we set $\rho(\mathcal{F}) = \mathcal{A}$.

The hypergraph $\mathcal{F}_{\{\bar{a}\}}$ contains two independent 2-element edges b_1b_2 and c_1c_2. Consequently, $k(\mathcal{F}_{\{\bar{a}\}}) \geq 2$. Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 3 & \text{if } A = \{a, \bar{b_1}, \bar{b_2}\}, \{a, \bar{c_1}, \bar{c_2}\} \\ 1 + 2\alpha & \text{if } A = \{\bar{a}\}. \end{cases}$$

The equation (4) specializes to

$$2\tau^{-3} + \tau^{-1-2\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.61$ and, consequently, $\tau_{\mathcal{F}} \leq 1.61$.

Case 15. The graph $\Gamma(a)$ has either one edge or two adjacent edges.

Let b be any vertex of $\Gamma(a)$, if $\Gamma(a)$ has one edge, or the vertex of degree 2, if $\Gamma(a)$ has two adjacent edges. We denote by c a neighbor of b in $\Gamma(a)$. We define

$$\mathcal{A} = \{\{\bar{a}\}, \{a, \bar{b}\}\}.$$

Every minimal transversal T satisfying $\{a\}$ satisfies $\{a, \bar{b}\}$ for otherwise $T - \{a\}$ would be a transversal too, contrary to the minimality of T. Hence, the family $\mathcal{A}$ is complete.

Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{\bar{a}\} \\ 2 & \text{if } A = \{a, \bar{b}\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1} + \tau^{-2} = 1.$$

We have $\tau'_{\mathcal{F}} \leq 1.62$ and, consequently, $\tau_{\mathcal{F}} \leq 1.62$.

Comment. It is easy to check that the only possibilities not covered by Cases 7 - 15 are when $\Gamma(a)$ is one of the following three graphs: the graph whose components are a triangle and a single edge (denoted by $C_3 \cup P_1$), the graph whose components are a 3-edge path and a single edge (denoted by $P_3 \cup P_1$), and the graph whose components are three single edges (denoted by $3P_1$).

Case 16. For every vertex a occurring in $\mathcal{F}$, $\Gamma(a)$ is isomorphic to $P_3 \cup P_1$, $C_3 \cup P_1$ or $3P_1$.

Subcase (i). There is a vertex a such that $\Gamma(a)$ is isomorphic to $P_3 \cup P_1$.

Let d, b, c and e be the consecutive vertices of the path P_3 in $\Gamma(a)$, and let f and g be the vertices of the isolated edge in $\Gamma(a)$.

Clearly, the graph $\Gamma(b)$ contains the edges ad and ac. Thus, it is not isomorphic to $3P_1$. Let us suppose that $\Gamma(b)$ is isomorphic to $C_3 \cup P_1$. Then, cd is an edge of $\Gamma(b)$ or, equivalently, $\mathcal{F}$ contains the edge bcd. Consequently, the graph $\Gamma(d)$ contains the edges ab and bc. Since a belongs to the following four edges in $\mathcal{F}$ only: abd, abc, ace and afg, no edge of the form ah, where $h \neq b$, is an edge of $\Gamma(d)$. It follows that $\Gamma(d)$ is isomorphic to $P_3 \cup P_1$ and that there is an edge ch in $\Gamma(d)$, for some $h \neq a, b$. Consequently, $\mathcal{F}$ contains the following edges: cdh, where $h \neq a, b$, ace, abc and bcd. These edges induce the edges ae, ab, bd and dh in $\Gamma(c)$. Since a, b, e, d and h are pairwise different, except for possibly $e = h$, $\Gamma(c)$ is isomorphic to either P_4 or C_4, a contradiction with the assumption of Case 16.

It follows that $\Gamma(b)$ is isomorphic to $P_3 \cup P_1$. Clearly, da and ac are edges of $\Gamma(b)$. Let b_1b_2, b_3b_4 be the remaining two edges of $\Gamma(b)$. Obviously, the edges b_1b_2 and b_3b_4 are independent and $b_1, b_2, b_3, b_4 \neq a$.

By symmetry, $\Gamma(c)$ is isomorphic to $P_3 \cup P_1$, too. Since it contains the edges ab and ae, it also contains an edge, say e_1e_2, with endvertices different form a, b and e.

Clearly, the collection $\mathcal{A}' = \{\{a, b, c\}, \{a, b, \bar{c}\}, \{a, \bar{b}\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}\}\}$ is complete. We define

$$\mathcal{A} = \{\{a, b, c, \bar{f}, \bar{g}\}, \{a, b, \bar{c}\}, \{a, \bar{b}\}, \{\bar{a}, b\}, \{\bar{a}, \bar{b}, c, d\}\}.$$

A standard argument shows that $\mathcal{A}$ is complete, as well (for instance, every minimal transversal T satisfying $\{a, b, c\}$ satisfies $\{a, b, c, \bar{f}, \bar{g}\}$ as otherwise $T - \{a\}$ would be a transversal, contradiction with minimality of T). Hence, the family $\mathcal{A}$ is complete and we define $\rho(\mathcal{F}) = \mathcal{A}$.

The hypergraph $\mathcal{F}_{\{a,b,\bar{c}\}}$ contains the 2-element edge e_1e_2. Thus, $k(\mathcal{F}_{\{a,b,\bar{c}\}}) \geq 1$. The hypergraph $\mathcal{F}_{\{a,\bar{b}\}}$ contains two independent 2-element edges b_1b_2 and b_3b_4 and so $k(\mathcal{F}_{\{a,\bar{b}\}}) \geq 2$. The hypergraph $\mathcal{F}_{\{\bar{a},b\}}$ contains two independent 2-element edges ce and fg, so $k(\mathcal{F}_{\{\bar{a},b\}}) \geq 2$. Finally, the hypergraph $\mathcal{F}_{\{\bar{a},\bar{b},c,d\}}$ contains the 2-element edge fg, so $k(\mathcal{F}_{\{\bar{a},\bar{b},c,d\}}) \geq 1$. It follows that

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 5 & \text{if } A = \{a, b, c, \bar{f}, \bar{g}\} \\ 3 + \alpha & \text{if } A = \{a, b, \bar{c}\} \\ 2 + 2\alpha & \text{if } A = \{a, \bar{b}\}, \{\bar{a}, b\} \\ 4 + \alpha & \text{if } A = \{\bar{a}, \bar{b}, c, d\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-5} + \tau^{-3-\alpha} + 2\tau^{-2-2\alpha} + \tau^{-4-\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.66$ and, consequently, $\tau_{\mathcal{F}} \leq 1.66$.

Subcase (ii). There is a vertex $a = a_1$ such that $\Gamma(a_1)$ is isomorphic to $C_3 \cup P_1$ and for no vertex x, $\Gamma(x)$ is isomorphic to $P_3 \cup P_1$.

Let a_2, a_3 and a_4 be the vertices of degree 2 in $\Gamma(a_1)$ and let b_1 and c_1 be the vertices of degree 1. Clearly, $b_1, c_1 \notin \{a_1, a_2, a_3, a_4\}$. Since the graph $\Gamma(a_2)$ has a vertex of degree 2 (the edges a_1a_3 and a_1a_4 belong to $\Gamma(a_2)$), the graph $\Gamma(a_2)$ is isomorphic to $C_3 \cup P_1$ and a_3a_4 is one of its edges, that is, $a_2a_3a_4$ is an edge in $\mathcal{F}$. Similarly, $\Gamma(a_3)$ and $\Gamma(a_4)$ are isomorphic to $C_3 \cup P_1$. For $i = 2, 3, 4$, let b_i and c_i be the vertices of degree 1 in $\Gamma(a_i)$. Clearly, $b_i, c_i \notin \{a_1, a_2, a_3, a_4\}$, for $i = 2, 3, 4$.

If, for some $i \neq j$, $\{b_i, c_i\} = \{b_j, c_j\}$, say $b_i = b_j$ and $c_i = c_j$, then pairs a_jc_i, c_ia_i are edges in $\Gamma(b_i)$ and the degree of c_i in $\Gamma(b_i)$ is 2. Hence, $\Gamma(b_i)$ is isomorphic to $C_3 \cup P_1$ and a_ia_j is an edge in $\Gamma(b_i)$, a contradiction (we note that $b_ia_ia_j$ is not a edge in $\mathcal{F}$ and so, b_ia_j is not an edge in $\Gamma(a_i)$).

Thus, $\{b_1, c_1\}$, $\{b_2, c_2\}$, $\{b_3, c_3\}$, $\{b_4, c_4\}$ are pairwise different. If each pair of the sets $\{b_i, c_i\}$, $i = 1, 2, 3, 4$, has a common vertex, then there is a vertex, say b_1, which belongs to all four sets. It follows that $a_ib_1c_i$, $i = 1, 2, 3, 4$, are edges in $\mathcal{F}$. Thus, $\Gamma(b_1)$ contains four different edges a_ic_i, $i = 1, 2, 3, 4$ and so, it is isomorphic to $C_3 \cup P_1$. Since a_1, a_2, a_3 and a_4 are pairwise different and $c_i \notin \{a_1, a_2, a_3, a_4\}$, for $i = 1, 2, 3, 4$, we get $c_1 = c_2 = c_3 = c_4$ ($\Gamma(b_1)$ has 5 vertices as it is isomorphic to $C_3 \cup P_1$). It follows that $\Gamma(b_1)$ has a vertex of degree 4, a contradiction.

Thus, some two of the sets $\{b_i, c_i\}$, $i = 1, 2, 3, 4$, are disjoint. Without loss of generality, $\{b_1, c_1\}$ and $\{b_2, c_2\}$ are disjoint. Clearly, it follows that b_1, c_1, b_2 and c_2 are pairwise different.

We define

$$\mathcal{A} = \{\{a_1\}, \{\bar{a_1}, a_2\}, \{\bar{a_1}, \bar{a_2}, a_3, a_4\}\}.$$

Since $\mathcal{A}' = \{\{a_1\}, \{\bar{a_1}, a_2\}, \{\bar{a_1}, \bar{a_2}\}\}$ is complete and every transversal satisfying $\{\bar{a_1}, \bar{a_2}\}$ satisfies $\{\bar{a_1}, \bar{a_2}, a_3, a_4\}$ (it intersects the edges $a_1a_2a_3$ and $a_1a_2a_4$), the family $\mathcal{A}$ is complete. We set $\rho(\mathcal{F}) = \mathcal{A}$.

The hypergraph $\mathcal{F}_{\{\bar{a_1}, a_2\}}$ contains two independent 2-element edges a_3a_4 (obtained from the 3-element edge $a_1a_3a_4$ in $\mathcal{F}$) and b_1c_1 (obtained from the 3-element edge $a_1b_1c_1$ in $\mathcal{F}$). Hence $k(\mathcal{F}_{\{\bar{a_1}, a_2\}}) \geq 2$. The hypergraph $\mathcal{F}_{\{\bar{a_1}, \bar{a_2}, a_3, a_4\}}$ also contains two disjoint 2-element edges: b_1c_1 (obtained from the 3-element edge $a_1b_1c_1$ in $\mathcal{F}$) and b_2c_2 (obtained from the 3-element edge $a_2b_2c_2$ in $\mathcal{F}$). Hence $k(\mathcal{F}_{\{\bar{a_1}, \bar{a_2}, a_3, a_4\}}) \geq 2$. Thus,

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 & \text{if } A = \{a_1\} \\ 2 + 2\alpha & \text{if } A = \{\bar{a_1}, a_2\} \\ 4 + 2\alpha & \text{if } A = \{\bar{a_1}, \bar{a_2}, a_3, a_4\}. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1} + \tau^{-2-2\alpha} + \tau^{-4-2\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.67$ and, consequently, $\tau_{\mathcal{F}} \leq 1.67$.

Subcase (iii). For every vertex a in $\mathcal{F}$ the graph $\Gamma(a)$ is isomorphic to $3P_1$.

Let b_1c_1, b_2c_2 and b_3c_3 be the edges of $\Gamma(a)$. We define

$$\mathcal{A} = \{\{\bar{a}\}, \{a, \bar{b_1}, \bar{c_1}\}, \{a, \bar{b_2}, \bar{c_2}\}, \{a, \bar{b_3}, \bar{c_3}\}\}.$$

It is a complete collection of conditions. Indeed, if T is a minimal transversal such that $a \in T$ and, for each $i = 1, 2, 3$, $b_i \in T$ or $c_i \in T$, then $T - \{a\}$ is a transversal, as well, a contradiction.

The hypergraph $\mathcal{F}_{\{\bar{a}\}}$ contains three pairwise disjoint 2-element edges b_1c_1, b_2c_2 and b_3c_3. Hence $k(\mathcal{F}_{\{\bar{a}\}}) \geq 3$. Since, for every $i = 1, 2, 3$, $\Gamma(b_i)$ consists of 3 independent edges one of which is ac_i, the hypergraph $\mathcal{F}_{\{a, \bar{b_i}, \bar{c_i}\}}$ contains two independent 2-element edges whose vertices are different from a and c_i. Hence, $k(\mathcal{F}_{\{a, \bar{b_i}, \bar{c_i}\}}) \geq 2$. It follows that

$$\Delta(\mathcal{F}, \mathcal{F}_A) \geq \begin{cases} 1 + 3\alpha & \text{if } A = \{\bar{a}\} \\ 3 + 2\alpha & \text{if } A = \{a, \bar{b_i}, \bar{c_i}\},\ i = 1, 2, 3. \end{cases}$$

The equation (4) specializes to

$$\tau^{-1-3\alpha} + 3\tau^{-3-2\alpha} = 1.$$

For $\alpha = 0.1950..$, we have $\tau'_{\mathcal{F}} \leq 1.66$ and, consequently, $\tau_{\mathcal{F}} \leq 1.66$.