Chapter 1

APPROXIMATIONS, STABLE OPERATORS, WELL-FOUNDED FIXPOINTS AND APPLICATIONS IN NONMONOTONIC REASONING

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Abstract In this paper we develop an algebraic framework for studying semantics of nonmonotonic logics. Our approach is formulated in the language of lattices, bilattices, operators and fixpoints. The goal is to describe fixpoints of an operator O defined on a lattice. The key intuition is that of an *approximation*, a pair (x, y) of lattice elements which can be viewed as an approximation to each lattice element z such that $x \leq z \leq y$. The key notion is that of an *approximating operator*, a monotone operator on the bilattice of approximations whose fixpoints approximate the fixpoints of the operator O. The main contribution of the paper is an algebraic construction which assigns a certain operator, called the *stable* operator, to every approximating operator on a bilattice of approximations. This construction leads to an abstract version of the well-founded semantics. In the paper we show that our theory offers a unified framework for semantic studies of logic programming, default logic and autoepistemic logic.

Keywords: Nonmonotonic logics, operators on lattices, fixpoints, approximating operators, well-founded fixpoint, stable fixpoints

1. INTRODUCTION

We study algebraic foundations of semantics of nonmonotonic knowledge representation formalisms. The algebraic framework we use is that of lattices, operators and fixpoints. The key tool is the theorem of Tarski and Knaster (Tarski, 1955) on fixpoints of monotone operators on complete lattices. Our work is motivated by the fact that all major semantics of knowledge representation formalisms such as logic programming, default logic and modal nonmonotonic logics are defined by means of fixpoints of suitably chosen operators on lattices of interpretations and possible-world structures. We derive general algebraic principles that lie behind these semantics.

Our work can be viewed as an extension of an abstract approach to logic programming proposed by Fitting. In a series of papers culminating in (Fitting, 1999), Fitting demonstrated that stable, supported, well-founded and Kripke-Kleene semantics of logic programs can be studied in abstract terms of fixpoints of two operators on a bilattice of 4-valued interpretations. One of these operators is the 4-valued van Emden-Kowalski operator $\mathcal{T}_P$ that generalizes a 2-valued van Emden-Kowalski operator T_P introduced in (van Emden and Kowalski, 1976). Fixpoints of the operator $\mathcal{T}_P$ yield the partial supported model semantics and Kripke-Kleene semantics for logic programs. The other operator is a 4-valued stable operator Ψ'_P introduced in (Przymusinski, 1990). The operator Ψ'_P can be regarded as a multi-valued generalization of the Gelfond-Lifschitz operator GL_P (Gelfond and Lifschitz, 1988). Fixpoints of the operator Ψ'_P determine the partial stable model semantics and the well-founded semantics.

In (Denecker et al., 1998; Denecker et al., 2000) we observed that an operator-based approach to logic programming put forth by Fitting can be adapted to the case of two other nonmonotonic systems: autoepistemic logic (Moore, 1984; Moore, 1985) and default logic (Reiter, 1980). In the case of autoepistemic logic, this abstract approach resulted in several new semantics. First, it allowed us to introduce for autoepistemic logic a counterpart to the semantics of extensions. Second, it led to generalizations of Kripke-Kleene and well-founded semantics. Most importantly, it exhibited the existence of a unifying framework behind all major semantics for autoepistemic logic. In the case of default logic, the operator-based approach led to a generalization of the Kripke-Kleene semantics and resulted in a uniform semantic framework for default logic, surprisingly similar to that discovered in the case of autoepistemic logic. In fact, in (Denecker et al., 2000) we proved that both frameworks are isomorphic and we argued that under the translation proposed in (Konolige, 1988), default logic can be viewed as a fragment of autoepistemic logic.

In this paper we extract essential algebraic elements underlying unified semantic frameworks for logic programming, autoepistemic logic and default logic developed in (Fitting, 1999; Denecker et al., 1998; Denecker et al., 2000). In the abstract setting we develop, we consider lattices, bilattices, operators on lattices and their approximations, that is, operators on bilattices. Elements of lattices represent some "points of interest". Operators describe ways in which one "point of interest" might be revised (updated) into another one. We are interested in fixpoints of operators on lattices as they are precisely those elements that cannot be revised away.

With each lattice we associate a certain bilattice (the product of the lattice by itself). The elements of such a bilattice can be interpreted as approximations to elements of the underlying lattice. To study fixpoints of an operator on a lattice, we introduce the concept of an approximating operator, defined on the associated bilattice. We demonstrate that studying fixpoints of approximating operators can provide us with insights into the structure and properties of fixpoints of operators they approximate. In particular, by considering all fixpoints of an approximating operator we obtain an abstract version of the Kripke-Kleene semantics. Adding some minimization requirements results in an abstract version of the well-founded semantics.

In knowledge representation applications "points of interest" represented by elements of lattices might be interpretations or possible-world structures describing truths (beliefs, knowledge) about a world specified by a base theory. Operators are formal descriptions of constraints on truth or belief sets used in revising one set of truths or beliefs into another one. We argue that our abstract setting yields as special cases semantic frameworks for logic programming, autoepistemic logic and default logic. We also show that all three systems exhibit an amazing similarity in the structure of the families of their semantics. By far the most important contribution of the paper is a general algebraic construction assigning to an arbitrary approximating operator its stable version. For each of the knowledge representation formalisms discussed here: logic programming, autoepistemic logic and autoepistemic logic, this construction allows us to reduce the study of all major semantics to the study of properties of a single operator.

Our work is concerned with abstract principles underlying nonmonotonic reasoning and with unified approaches to nonmonotonicity. In this respect it is somewhat similar to the work by Bochman (Bochman, 1996; Bochman, 1998a; Bochman, 1998b), and by Brass and Dix (Brass and Dix, 1999). Bochman develops an abstract proof-theoretic approach to nonmonotonicity based on the notion of a biconsequence relation. Brass and Dix characterize semantics for nonmonotonic systems in terms of general abstract postulates on their properties.

The paper is organized as follows. In Section 2. we briefly review key concepts and definitions related to lattices, bilattices and operators on them. In Section 3. we formally introduce the notion of an approximating operator and establish a number of basic properties of these operators. We also discuss there an abstract version of the Kripke-Kleene semantics. Next, in Section 4. for every approximating operator we define its stable operator and an abstract form of the well-founded semantics. We discuss applications of our approach in knowledge representation in Section 5. The last section contains conclusions, open problems and a discussion of future work.

2. PRELIMINARIES FROM LATTICE THEORY

A *lattice* is a partially ordered set $\langle L, \leq \rangle$ such that every two element set $\{x, y\} \subseteq L$ has a *least upper bound*, $lub(x, y)$, and a *greatest lower bound*, $glb(x, y)$. A lattice $\langle L, \leq \rangle$ is *complete* if every subset of L has both least upper and greatest lower bounds. Consequently, a complete lattice has a least element ($\bot$) and a greatest element ($\top$).

An *operator* on a lattice $\langle L, \leq \rangle$ is any function from L to L. An operator O on L is *monotone* if for every pair of elements $x, y \in L$,

$$x \leq y \text{ implies } O(x) \leq O(y).$$

Similarly, an operator O on L is *antimonotone* if for every pair x, y of elements from L,

$$x \leq y \text{ implies } O(y) \leq O(x).$$

The composition of two antimonotone operators is monotone, as stated in the following result.

Proposition 1 *If the operators $O_1 : L \to L$, $O_2 : L \to L$ are antimonotone, then the operator $O_1 \circ O_2$ is monotone.*

Another straightforward observation asserts that operators that are both monotone and antimonotone are constant.

Proposition 2 *If an operator $O : L \to L$ is monotone and antimonotone then it is constant.*

The basic tool to study fixpoints of operators on lattices is the celebrated theorem by Tarski and Knaster (Tarski, 1955).

Theorem 3 *Let O be a monotone operator on a complete lattice $\langle L, \leq \rangle$. Then, O has a fixpoint and the set of all fixpoints of O is a complete lattice. The least fixpoint of this lattice (that is, the least fixpoint of O) can be obtained by iterating O over $\bot$. The greatest fixpoint of this lattice (the greatest fixpoint of O) can be obtained by iterating O over $\top$.*

We denote the least and the greatest fixpoints of the operator O by $lfp(O)$ and $gfp(O)$, respectively.

In applications it is often useful, and sometimes necessary, to approximate elements of lattices. We say that an element $z \in L$ is approximated by a pair $(x, y) \in L^2$ if $x \leq z \leq y$. Approximations of the form (x, x) are especially interesting. They provide a complete description of an element they approximate and so, we refer to them as *complete*. There is a straightforward one-to-one correspondence between L and the set of complete elements of L^2.

Since approximations are the key concept of our approach, in the paper, we study the set L^2, operators on L^2 and fixpoints of these operators.

The set L^2 can be endowed with two natural orderings. The first of them is a generalization of an ordering $\leq$ from L. We will refer to it as the *lattice* ordering and use the same symbol $\leq$ to denote it. Formally, it is defined by

$$(x, y) \leq (x_1, y_1) \text{ if } x \leq x_1 \text{ and } y \leq y_1.$$

The second ordering, called the *information* ordering, captures the intuition of increased precision of the approximation. This ordering, denoted $\leq_i$, is defined by

$$(x, y) \leq_i (x_1, y_1) \text{ if } x \leq x_1 \text{ and } y_1 \leq y.$$

It is easy to see that L^2 with each of these two orderings induces a complete lattice. In addition, it can be shown that the twelve distributivity laws involving the meets and joins with respect to both orderings all hold. Such algebraic structures are known as *bilattices* (Ginsberg, 1988; Fitting, 1999). They were used by Fitting in his discussion of semantics of logic programs with negation.

Not all pairs $(x, y) \in L^2$ can be interpreted as approximations to elements of L. For that to be the case, it is necessary that $x \leq y$. Thus, we say that a pair $(x, y) \in L^2$ is *consistent* if $x \leq y$. Otherwise, it is called *inconsistent*. Clearly, consistent pairs can be viewed as descriptions of our, in general, incomplete knowledge about elements from L that they approximate. Inconsistent pairs can be viewed as describing the fact that our knowledge about some unknown elements from L is inconsistent. The information ordering when applied to inconsistent pairs can be regarded as an ordering measuring the "degree of inconsistency".

Clearly, the collection of consistent pairs does not form a sublattice of L^2. Indeed, each element of the form (x, x) is a maximal consistent element of L^2. Thus, no two different elements of the form (x, x) have a consistent upper bound. By allowing inconsistent approximations into our considerations we get an intuitive duality between consistent and inconsistent pairs, and between the degree of precision and the degree of inconsistency. We deal with a much richer algebraic structure and obtain a more elegant theory. In the same time, all main constructions described in the paper are, in fact, restricted to the consistent part of a bilattice of approximations and both the Kripke-Kleene and well-founded fixpoints, that we define later, are consistent (however, dual constructions for the inconsistent part of the bilattice can also be considered).

The theorem by Tarski and Knaster talks about fixpoints of monotone operators. It implies also some important properties of antimonotone operators. A pair of elements $x, y \in L$ is an *oscillating pair* an operator O on L if $y = O(x)$ and $x = O(y)$. In other words, x and y form an oscillating pair if and only if x is a fixpoint of $O^2 = O \circ O$ and $y = O(x)$. An oscillating pair (x, y) is an *extreme* oscillating pair for O if for every oscillating pair (x', y') for O, $(x, y) \leq_i (x', y')$ and $(x, y) \leq_i (y', x')$ (or equivalently, $x \leq x', y' \leq y$). In particular, if (x, y) is an extreme oscillating pair then $x \leq y$. It is also easy to see that if an extreme oscillating pair exists, it is unique.

Theorem 4 *Let O be an antimonotone operator on a complete lattice $\langle L, \leq \rangle$. Then, O^2 has a least fixpoint and a greatest fixpoint and $(lfp(O^2), gfp(O^2))$ is the unique extreme oscillating pair of O.*

In this paper, we study fixpoints of operators on lattices by considering fixpoints of associated operators on bilattices. These operators often satisfy some monotonicity properties. Thus, in the remainder of this section, we present results on operators on L^2 that are monotone or antimonotone with respect to the orderings $\leq$ and $\leq_i$. Before we present our results, we need more terminology.

Let us consider an operator A on L^2. Let us denote by A^1 and A^2 the functions from L^2 to L such that

$$A(x, y) = (A^1(x, y), A^2(x, y)).$$

We say that A is *symmetric* if $A^1(x, y) = A^2(y, x)$. Clearly, if an operator $A : L^2 \to L^2$ is symmetric then for every $x \in L$, $A^1(x, x) = A^2(x, x)$.

In our discussion in the remainder of this paper we will restrict our considerations to symmetric operators only. The motivation for this restriction is twofold. First, all operators that appear in knowledge representation applications (for instance, the 4-valued van Emden-Kowalski operator $\mathcal{T}_P$) are symmetric. Second, the assumption of symmetry results in a much more elegant theory. In particular, symmetric operators are *extending*, an important property in our theory of approximations (we introduce this notion in the next section). However, we stress that the assumption of symmetry is not essential and all major concepts and constructions described in the paper can be developed without it.

Proposition 5 *A symmetric operator $A : L^2 \to L^2$ is $\leq_i$-monotone if and only if for every $y \in L$, $A^1(\cdot, y)$ is monotone and for every $x \in L$, $A^1(x, \cdot)$ is antimonotone (or equivalently, if and only if for every $y \in L$, $A^2(\cdot, y)$ is antimonotone and for every $x \in L$, $A^2(x, \cdot)$ is monotone).*

The next result provides a similar characterization of all symmetric operators on L^2 that are monotone with respect to the ordering $\leq$.

Proposition 6 *A symmetric operator $A : L^2 \to L^2$ is $\leq$-monotone if and only if for every $x, y \in L$, $A^1(x, \cdot)$ and $A^1(\cdot, y)$ are monotone (or, equivalently, if and only if for every $x, y \in L$, $A^2(x, \cdot)$ and $A^2(\cdot, y)$ are monotone).*

Propositions 5 and 6, together with Proposition 2, imply a characterization of symmetric operators that are both $\leq_i$-monotone and $\leq$-monotone.

Proposition 7 *An operator $A : L^2 \to L^2$ is symmetric and monotone with respect to both $\leq_i$ and $\leq$ if and only if there is a monotone operator $O : L \to L$ such that for every $x, y \in L$, $A(x, y) = (O(x), O(y))$.*

Next, we present a description of symmetric operators on L^2 that are $\leq_i$-monotone and $\leq$-antimonotone.

Proposition 8 *An operator $A : L^2 \to L^2$ is symmetric, $\leq_i$-monotone and $\leq$-antimonotone if and only if there is an antimonotone operator $O : L \to L$ such that for every $x, y \in L$, $A(x, y) = (O(y), O(x))$.*

Propositions 7 and 8 imply that there is a one-to-one correspondence between monotone (antimonotone, respectively) operators on L and $\leq_i$-monotone and $\leq$-monotone ($\leq_i$-monotone and $\leq$-antimonotone, respectively) operators on L^2.

When L is a complete lattice, it follows by Knaster-Tarski Theorem and by Theorem 4 that an $\leq_i$-monotone and $\leq$-antimonotone operator

$A : L^2 \rightarrow L^2$ has $\leq_i$-least and $\leq_i$-greatest fixpoints and a $\leq$-extreme oscillating pair. Let us denote the $\leq_i$-least fixpoint of A by q_A, and the $\leq_i$-greatest fixpoint of A by Q_A. Similarly, let us denote the $\leq$-extreme oscillating pair for A by (e_A, E_A).

If $A : L^2 \rightarrow L^2$ is, in addition, symmetric, by Proposition 8, there is an antimonotone operator $O : L \rightarrow L$ such that $A(x,y) = (O(y), O(x))$. Let us denote by q the least fixpoint of O^2 and by Q the greatest fixpoint of O^2 (Tarski-Knaster Theorem applies as O^2 is monotone). The following theorem, due essentially to Fitting, summarizes the relations between the fixpoints and extreme pairs defined above.

Theorem 9 *Let L be a complete lattice. Let $A : L^2 \rightarrow L^2$ be a symmetric $\leq_i$-monotone and $\leq$-antimonotone operator on L^2. Then:*

1. $q_A = (q,Q)$, $Q_A = (Q,q)$, $e_A = (q,q)$, $E_A = (Q,Q)$
2. $q_A = glb_{\leq_i}(e_A, E_A)$ *and* $Q_A = lub_{\leq_i}(e_A, E_A)$
3. $e_A = glb_{\leq}(q_A, Q_A)$ *and* $E_A = lub_{\leq}(q_A, Q_A)$.

Proof: Let $O : L \rightarrow L$ be an antimonotone operator such that $A(x,y) = (O(y), O(x))$ (Proposition 8) and let q and Q be the least and the greatest fixpoints of O^2, respectively. Then, (q,Q) is the extreme oscillating pair of O (Theorem 4), $O(q) = Q$ and $O(Q) = q$. Thus, $A(q,Q) = (O(Q), O(q)) = (q,Q)$ or, equivalently, (q,Q) is a fixpoint of A. Let (x,y) be a fixpoint of A. Then, $(x,y) = A(x,y) = (O(y), O(x))$ and $x = O(y)$ and $y = O(x)$. Thus, (x,y) is an oscillating pair for O. Since (q,Q) is the extreme oscillating pair for O, $(q,Q) \leq_i (x,y)$. It follows that (q,Q) is the least fixpoint of A or, in other words, that $q_A = (q,Q)$. The proof that $Q_A = (Q,q)$ is similar.

Next, observe that $A(q,q) = (O(q), O(q)) = (Q,Q)$ and $A(Q,Q) = (O(Q), O(Q)) = (q,q)$. Thus, $((q,q),(Q,Q))$ is an oscillating pair for A. Let $((x,y),(x',y'))$ be an oscillating pair for A. Then, (x,y) and (x',y') are fixpoints of A^2. Consequently, x, y, x' and y' are all fixpoints of O^2. It follows that $q \leq x, y, x', y' \leq Q$ and so, $(q,q) \leq_i (x,y), (x',y') \leq_i (Q,Q)$. Thus, $((q,q),(Q,Q))$ is the extreme oscillating pair for A (or, equivalently, if $e_A = (q,q)$ and $E_A = (Q,Q)$).

The assertions (2) and (3) follow immediately from the assertion (1) and the fact that $q \leq Q$. □

3. APPROXIMATING OPERATORS

Our paper is an attempt to identify basic algebraic principles behind semantics of nonmonotonic reasoning formalisms. The key concept to our approach is that of an approximating operator. Given an operator O on a lattice L the goal is to gain insights into its fixpoints and into constructive techniques to find them. To this end, we will consider operators on the bilattice L^2.

Definition 10 *An operator $A : L^2 \rightarrow L^2$* extends *an operator $O : L \rightarrow L$ if for every $x \in L$, $A(x,x) = (O(x), O(x))$. An operator $A : L^2 \rightarrow L^2$ is* extending *if for every $x \in L$, there is $y \in L$ such that $A(x,x) = (y,y)$.*

We define the *diagonal* of L^2 to be the set $\{(x,x) : x \in L\}$ (that is, the set of all complete approximations). If an operator $A : L^2 \rightarrow L^2$ extends $O : L \rightarrow L$ then the behavior of A on the diagonal fully describes the behavior of O. In particular, complete fixpoints of A correspond to fixpoints of O.

Proposition 11 *Let O be an operator on a lattice L and let A be an operator on L^2 extending O. Then, x is a fixpoint of O if and only if (x,x) is a fixpoint of A.*

If A is symmetric then for each lattice element x, $A^1(x,x) = A^2(x,x)$. Hence $A(x,x)$ is complete and, consequently, A is extending. This observation is stated in the following result. As we mentioned earlier, it is one of the motivations for restricting our discussion to symmetric operators only.

Proposition 12 *If an operator $A : L^2 \rightarrow L^2$ is symmetric then A is extending.*

It follows directly from the definition of an extending operator that to study fixpoints of an operator O one might construct an appropriate extending operator A and study its fixpoints instead. Clearly, complete fixpoints of the operator A would then provide a complete description of the fixpoints of O.

It seems that this new problem is essentially the same as the original one. There is, however, one difference. An extending operator A is defined on a bilattice. Consequently, all its fixpoints are approximated by the least element $(\bot, \top)$ of the bilattice (referred to as the *weakest approximation*). Two natural questions arise: are there better approximations to fixpoints of A than this trivial one, and can they be constructed. In general the answer is negative. However, the answer is positive if A is $\leq_i$-monotone. In such case, we can iterate A starting with the weakest approximation. In each iteration we improve the precision of the approximation. When no further improvement is possible the process terminates and results in the $\leq_i$-least fixpoint of A. This fixpoint approximates all fixpoints of A, it is often better than the weakest approximation $(\bot, \top)$ and it can be constructed! The possibility of constructing the least fixpoint of a $\leq_i$-monotone extending operator leads us to one of the key concepts of the paper (in view of our remarks, we introduce it with the stronger requirement of symmetry).

Definition 13 *An operator $A : L^2 \rightarrow L^2$* approximates *an operator $O : L \rightarrow L$ if A is symmetric, extends O and is $\leq_i$-monotone. An operator $A : L^2 \rightarrow L^2$ is* approximating *if it is symmetric and $\leq_i$-monotone.*

We say that an operator $A : L^2 \rightarrow L^2$ is *consistent* if it maps consistent pairs to consistent pairs, that is whenever (x, y) is consistent, then also $A(x, y)$ is consistent. The following two results formally state basic properties of approximating operators.

Proposition 14 *If $A : L^2 \rightarrow L^2$ is an approximating operator, then A is consistent.*

Corollary 15 *Let $A : L^2 \rightarrow L^2$ be an approximating operator for an operator $O : L \rightarrow L$. Then, A has a $\leq_i$-least fixpoint. This fixpoint is consistent and approximates every fixpoint of O.*

The notion of $\leq_i$-least fixpoint of an operator A approximating operator O in lattice L is an important concept. The least fixpoint of A approximates all fixpoints of O. Speaking informally, it determines information that is common to all the fixpoints of O. Next, if the $\leq_i$-least fixpoint is complete, say it is of the form (x, x), then x is the only fixpoint of O. Moreover, in such case, this unique fixpoint of O is based on a constructive principle of building it incrementally by iterating the approximating operator A. Since in the case of logic programming, the concept of the $\leq_i$-least fixpoint of an approximating operator can be specialized to the Kripke-Kleene semantics, we refer to the $\leq_i$-least fixpoint of an approximating operator A as the *Kripke-Kleene fixpoint* of A. We denote this fixpoint by α_A.

Clearly, an operator O on a lattice may have several approximating operators. Each gives rise to its Kripke-Kleene fixpoint and the corresponding approximation of all fixpoints of O. The problem of finding an approximation operator providing the best (in some sense) approximation is, in general, a challenging one. We do have some results that pertain to it. They will be a subject of another paper. Here we will only mention two simple special cases when an operator O is monotone or antimonotone.

Let O be a monotone operator on L. By Proposition 7, the operator $A_O(x, y) = (O(x), O(y))$ is $\leq_i$-monotone. It is also symmetric, consistent and extends the operator O. Hence, A_O is an approximating operator for O. By Proposition 7, A_O is $\leq$-monotone. In fact, Proposition 7 implies that A_O is a unique approximating operator for O that is $\leq$-monotone. The least $\leq_i$-fixpoint of A_O is $(lfp(O), lfp(O))$. We will call A_O the *trivial* approximating operator for a monotone operator O[1].

Similarly, if A is an antimonotone operator on L then, by Proposition 8, the operator $A_O(x,y) = (O(y), O(x))$ is $\leq_i$-monotone. In addition, A_O is symmetric, consistent and it extends O. Hence, it is an approximating operator for O. By Proposition 8, A_O is $\leq$-antimonotone and, in fact, it is a unique approximating operator for O that is $\leq$-antimonotone. We will call A_O the *trivial* approximating operator for an antimonotone operator O. Theorem 9 characterizes the fixpoints and the extreme oscillating pair of the trivial approximating operator for an antimonotone operator O.

4. STABLE OPERATOR AND WELL-FOUNDED FIXPOINT

In the case of logic programming, fixpoints of the van Emden-Kowalski operator T_P determine (2-valued) supported models of a program P. Supported model semantics (also known as Clark completion semantics) is often too weak for knowledge representation applications. The class of *stable* models was proposed in (Gelfond and Lifschitz, 1988) as the basis of an alternative semantics for programs with negation.

It is well-known that stable models form a subclass of the class of supported models. Our goal in this section is to study abstract principles relating supported and stable models. More generally, we search for principles that might allow us to identify interesting special subclasses in the class of all fixpoints of an operator O defined on a complete lattice L. Since, as argued in the previous section, fixpoints of O can be studied by considering approximating operators, our approach is to search for principles that allow us to narrow down the class of fixpoints of approximating operators. Approximating operators are symmetric and $\leq_i$-monotone. The results in this section rely only on these two assumptions (however, as mentioned earlier, the assumption of symmetry is not essential for our theory).

The fact that bilattices are also ordered by the (generalization of) lattice ordering suggests a possible approach. Minimizing truth is the key idea underlying commonsense reasoning and the process of jumping

[1]This algebraic property of monotone operators explains why all major nonmonotonic semantics coincide on the class of Horn theories (or programs) and are given by the least fixpoint construction.

to conclusions. In our abstract setting, it boils down to minimization with respect to $\leq$ and we focus our attention on those fixpoints of A which are $\leq$-minimal. However, the principle of $\leq$-minimality is in itself not sufficient. For instance, it is well known that not every minimal supported model of a logic program P is stable.

In this section we describe an algebraic construction that assigns to every $\leq_i$-monotone operator A on a bilattice L^2 its *stable* operator $\mathcal{C}_A$ defined also on L^2. We demonstrate that every fixpoint of the operator $\mathcal{C}_A$ is a $\leq$-minimal fixpoint of A. Later in the paper we argue that fixpoints of stable operators appear naturally in several nonmonotonic reasoning formalisms such as logic programming, default logic and autoepistemic logics, thus validating our construction.

Definition 16 *Let L be a complete lattice. Let an operator $A : L^2 \to L^2$ on a bilattice L^2 be symmetric and $\leq_i$-monotone.*

1. *The* complete stable operator for A, *$C_A : L \to L$, is defined by $C_A(y) = \mathit{lfp}(A^1(\cdot, y))$ (or, equivalently, by, $C_A(y) = \mathit{lfp}(A^2(y, \cdot))$).*
2. *The* stable operator for A, *$\mathcal{C}_A : L^2 \to L^2$ is defined by $\mathcal{C}_A(x, y) = (C_A(y), C_A(x))$.*

Since for every $y \in L$ the operators $A^1(\cdot, y)$ and $A^2(y, \cdot)$ are monotone (Proposition 5), the operators C_A and $\mathcal{C}_A$ are well-defined.

The intuition behind the stable operator is as follows. We are given an operator $A : L^2 \to L^2$. This operator can be viewed as a description of a way to revise approximations (x, y). Our goal is to derive from A a different (but related) way to "revise" approximations. We proceed as follows. Given an approximation (x, y), to construct the lower bound of a new approximation we use y — our *current* upper estimate. We consider the operator $A^1(\cdot, y)$ which models revisions of the lower bounds of those approximations with the upper bound fixed to y. Since $A^1(\cdot, y)$ is a monotone operator, there is a natural candidate for the intended new lower bound — the least fixpoint of $A^1(\cdot, y)$. To construct the new upper bound, we proceed similarly. We use the current lower bound x and consider the operator $A^2(x, \cdot)$. This operator is monotone and its least fixpoint is selected as the new intended upper bound. Since A is symmetric, the same operator, C_A, can be used to determine both the lower and the upper bound.

Let us consider an operator A that is both $\leq_i$- and $\leq$-monotone. Such operators are described in Proposition 7. They are of the form $A(x, y) = (O(x), O(y))$, where O is monotone. It follows that $C_A(y) = \mathit{lfp}(O)$ and does not depend on y. Thus, we get the following result.

Proposition 17 *Let L be a complete lattice. Let $A : L^2 \rightarrow L^2$ be an operator monotone with respect to $\leq_i$ and $\leq$. Then $\mathcal{C}_A$ is constant.*

If an operator A is $\leq_i$-monotone and $\leq$-antimonotone then, by Proposition 8, there is an antimonotone operator O such that $A(x, y) = (O(y), O(x))$. Consequently, $A(\cdot, y) = O(y)$. It follows that $C_A(y) = O(y)$, that is, the stable operator for the operator A is A itself.

Proposition 18 *Let L be a complete lattice. Let $A : L^2 \rightarrow L^2$ be an operator monotone with respect to $\leq_i$ and antimonotone with respect to $\leq$. Then $\mathcal{C}_A = A$.*

We will now study properties of the stable operator $\mathcal{C}_A$ and its fixpoints. Our first result shows that fixpoints of $\mathcal{C}_A$ are $\leq$-minimal fixpoints of A (the converse statement in general does not hold).

Theorem 19 *Let L be a complete lattice. Let an operator $A : L^2 \rightarrow L^2$ on a bilattice L^2 be $\leq_i$-monotone. Every fixpoint of the stable operator $\mathcal{C}_A$ is a $\leq$-minimal fixpoint of A.*

Proof: In this proof we will use some additional basic properties of operators on lattices. An element x of a lattice L is a *pre-fixpoint* of an operator $O : L \rightarrow L$ if $O(x) \leq x$. The argument of Tarski and Knaster shows that if L is a complete lattice and O is a monotone operator on L then for every pre-fixpoint x of O, $lfp(O) \leq x$.

Let (x, y) be a fixpoint of $\mathcal{C}_A$. It follows that $(x, y) = (C_A(y), C_A(x))$. By the definition of C_A, $x = lfp(A^1(\cdot, y))$, and hence $A^1(x, y) = x$. Similarly, $y = lfp(A^1(\cdot, x)) = lfp(A^2(x, \cdot))$. Thus, $A^2(x, y) = y$. Consequently, (x, y) is a fixpoint of A.

Next, assume that (x', y') is a fixpoint of A such that $(x', y') \leq (x, y)$. It follows that $x' \leq x$ and hence, by antimonotonicity of $A^2(\cdot, y')$ (Proposition 5), we have that $A^2(x, y') \leq A^2(x', y') = y'$. Thus, y' is a pre-fixpoint of the operator $A^2(x, \cdot)$. Since $A^2(x, \cdot)$ is monotone, and y is its least fixpoint, it follows that $y \leq y'$. Since $(x', y') \leq (x, y)$, $y = y'$. Similarly, one can derive that $x = x'$. Thus, $(x', y') = (x, y)$ which, in turn, implies that (x, y) is a $\leq$-minimal fixpoint of A. □

Theorem 19 shows, in particular, that if A is $\leq_i$-monotone, a fixpoint of $\mathcal{C}_A$ is also a fixpoint of A. We will call every fixpoint of the stable operator $\mathcal{C}_A$ a *stable* fixpoint of A.

Directly from the definition of the operators C_A and from Proposition 5 it follows that C_A is antimonotone. Consequently, by Proposition 8, $\mathcal{C}_A$ is $\leq_i$-monotone and $\leq$-antimonotone.

Proposition 20 *Let L be a complete lattice. Let A be a symmetric $\leq_i$-monotone operator on L^2. Then, C_A is an antimonotone operator on L and $\mathcal{C}_A$ is a $\leq_i$-monotone and $\leq$-antimonotone operator on L^2.*

Propositions 18 and 20 imply the following corollary that states that applying the stability construction to a stable operator does not lead to a new operator anymore.

Corollary 21 *Let L be a complete lattice. Let A be a symmetric $\leq_i$-monotone operator on L^2. Then $\mathcal{C}_{\mathcal{C}_A} = \mathcal{C}_A$.*

It is also easy to see that $\mathcal{C}_A$ is symmetric and extends the operator C_A. Thus, we obtain the following corollary to Proposition 20.

Corollary 22 *Let L be a complete lattice. Let A be a $\leq_i$-monotone operator on L^2. Then, the stable operator $\mathcal{C}_A$ is a trivial approximation of the complete stable operator C_A.*

Since $\mathcal{C}_A$ is $\leq_i$-monotone and $\leq$-antimonotone, it has a $\leq_i$-least fixpoint, a $\leq_i$-greatest fixpoint and also a $\leq$-extreme oscillating pair. As explained in Theorem 9, these concepts are interrelated and can be expressed in terms of the fixpoints of the operator $C_A^2 = C_A \circ C_A$.

The $\leq_i$-least fixpoint of $\mathcal{C}_A$ is of particular interest as it provides an approximation to every stable fixpoint of A. We call the $\leq_i$-least fixpoint of $\mathcal{C}_A$ the *well-founded fixpoint* of a $\leq_i$-monotone operator A and denote it by β_A. The choice of the term is dictated by the fact that in the case of logic programming, the least fixpoint of the stable operator for the 4-valued van Emden-Kowalski operator $\mathcal{T}_P$ yields the well-founded semantics.

The following result gathers several properties of the well-founded fixpoint of an operator that generalize properties of the well-founded model of a logic program.

Theorem 23 *Let L be a complete lattice. Let $A : L^2 \to L^2$ be a $\leq_i$-monotone symmetric operator.*

1. *The Kripke-Kleene fixpoint α_A and the well-founded fixpoint β_A satisfy $\alpha_A \leq_i \beta_A$*
2. *For every stable fixpoint x of A, $\beta_A \leq_i x$*
3. *If β_A is complete then it is the only consistent stable fixpoint of A.*
4. *The operator $\mathcal{C}_A$ is consistent and, consequently, β_A is consistent, too.*

Proof: The assertion (1) follows from the fact that α_A is the $\leq_i$-least fixpoint of A and fixpoints of $\mathcal{C}_A$ are fixpoints of A (Theorem 19).

Stable fixpoints of A are precisely the fixpoints of $\mathcal{C}_A$. Since β_A is the least fixpoint of $\mathcal{C}_A$, the assertion (2) follows.

To prove (3), we first observe that since β_A is complete, it is a consistent stable fixpoint of A. Let us consider a consistent stable fixpoint of A, say x. Then x is a fixpoint of $\mathcal{C}_A$. Thus, $\beta_A \leq_i x$. Since β_A is complete, it is a maximal consistent element of L^2. Thus, $x = \beta_A$ and (3) follows.

Finally, $\mathcal{C}_A$ is an approximating operator (it approximates operator C_A. Thus, the assertion (4) follows from Proposition 14 and Corollary 15. □

We will now assume that A is an approximating operator for an operator $O : L \rightarrow L$ and discuss the relationship between the fixpoints of $\mathcal{C}_A$ and fixpoints of O.

Proposition 24 *Let L be a complete lattice. Let $A\colon L^2 \rightarrow L^2$ be an approximating operator for an operator $O\colon L \rightarrow L$. If (x, x) is a fixpoint of $\mathcal{C}_A$ then x is a $\leq$-minimal fixpoint of O.*

Proof: The proposition follows immediately from theorem 19. □

It follows from Proposition 24 that if A is an approximating operator for an operator O then fixpoints of O corresponding to complete fixpoints of the stable operator $\mathcal{C}_A$ form an antichain.

We will next consider the case when O is monotone. In this case we can use the trivial approximation of O, A_O. Using Proposition 17 and the discussion that precedes it, we obtain the following result.

Proposition 25 *Let L be a complete lattice. If $O : L \rightarrow L$ is a monotone operator, then for every $x \in L$, $\mathcal{C}_{A_O}(x, y) = (lfp(O), lfp(O))$ (that is, C_{A_O} is constant).*

If O is monotone, its trivial approximation A_O may have many fixpoints in general and many complete fixpoints, in particular. However, by Proposition 25, the stable operator for A_O has only one fixpoint and it corresponds precisely to the least fixpoint of O. In the context of logic programming, this result says that a Horn logic program P has a unique stable model and that it coincides with the least Herbrand model of P.

Consider an operator O defined on a complete lattice L. How can we associate with this operator its well-founded fixpoint? In order to do so, we need to construct an approximation A of O and use the well founded fixpoint of A as the well-founded fixpoint of O. There may be several approximating operators and the well-founded fixpoints of these operators may have different properties. As mentioned earlier, a study of best approximations will be presented in another paper.

5. APPLICATIONS IN KNOWLEDGE REPRESENTATION

The results presented here provide us with a uniform framework for semantic studies of major knowledge representation formalisms: logic programming, autoepistemic logic and default logic. Namely, all major semantics for each of these formalisms can be derived from a single operator.

In the case of logic programming, our results extend an algebraic approach proposed in (Fitting, 1999). The lattice of interest here is that of 2-valued interpretations of the Herbrand base of a given program P. We will denote it by $\mathcal{A}_2$. The corresponding bilattice $\mathcal{A}_2 \times \mathcal{A}_2$ is isomorphic with the bilattice $\mathcal{A}_4$ of 4-valued interpretations (in 4-valued Belnap logic). Our results imply that the central role in logic programming is played by the 4-valued van Emden-Kowalski operator $\mathcal{T}_P$ defined on the bilattice $\mathcal{A}_2 \times \mathcal{A}_2$ (or, equivalently, on bilattice $\mathcal{A}_4$). First, the operator $\mathcal{T}_P$ approximates the 2-valued van Emden-Kowalski operator T_P. Second, fixpoints of $\mathcal{T}_P$ represent 4-valued supported models, consistent fixpoints of $\mathcal{T}_P$ represent partial (3-valued) supported models and complete fixpoints of $\mathcal{T}_P$ describe supported models of P. The $\leq_i$-least fixpoint of $\mathcal{T}_P$ (it exists as $\mathcal{T}_P$ is approximating) defines the Kripke-Kleene semantics of P.

Perhaps most importantly, it turns out that our general construction assigning the stable operator to every approximating operator when applied to $\mathcal{T}_P$ yields the 4-valued Przymusinski operator Ψ'_P and the 2-valued Gelfond-Lifschitz operator GL_P. That is, the stable operator for $\mathcal{T}_P$ coincides with Ψ'_P and the complete stable operator for $\mathcal{T}_P$ coincides with GL_P. Thus, the semantics of 4-valued, partial (3-valued) and 2-valued stable models can also be derived from the operator $\mathcal{T}_P$. The same is true for the well-founded semantics since it is determined by the $\leq_i$-least fixpoint of the stable operator of $\mathcal{T}_P$. The structure of the family of operators and semantics for logic programming that can be derived from the operator $\mathcal{T}_P$ is presented in Figure 1.1.

In (Denecker et al., 1998; Denecker et al., 2000) we developed an algebraic approach to semantics for autoepistemic and default logics. In both cases, our approach can be regarded as a special case of the general approach presented here. In the investigations of autoepistemic and default logics we consider the lattice $\mathcal{W}$ of possible-world structures (sets of 2-valued interpretations) and the corresponding bilattice $\mathcal{B}$ of

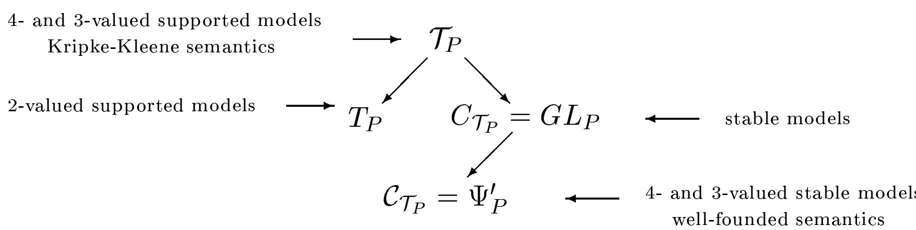

Figure 1.1 Operators and semantics associated with logic programming

belief pairs (Denecker et al., 1998). In the case of autoepistemic logic, the central place is occupied by the operator $\mathcal{D}_T$ (T is a given modal theory) defined on the bilattice of belief pairs and introduced in (Denecker et al., 1998). It turns out to be an approximating operator for the operator D_T used by Moore to define the notion of an expansion (Moore, 1984). Thus, the concepts of partial expansions and expansions can be derived from $\mathcal{D}_T$. Similarly, the Kripke-Kleene semantics can be obtained from $\mathcal{D}_T$ as its least fixpoint. The stable operator for $\mathcal{D}_T$ and its complete counterpart lead to semantics for autoepistemic logic that to the best of our knowledge have not been studied in the literature: the semantics of extensions, partial extensions and the well-founded semantics, that are closely related to the corresponding semantics for default logic (Denecker et al., 2000). The emerging structure of operators and semantics for autoepistemic logic is depicted in Figure 1.2.

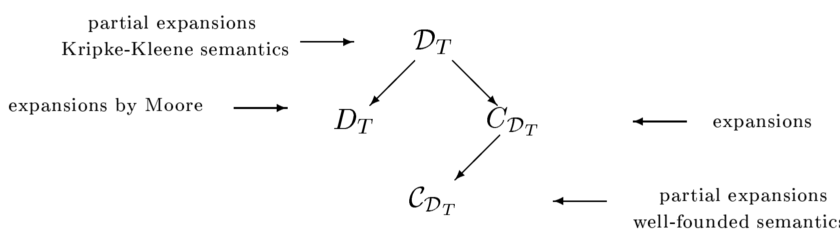

Figure 1.2 Operators and semantics associated with autoepistemic logic

A very similar picture emerges in the case of default logic, too. In (Denecker et al., 2000) we described an operator $\mathcal{E}_\Delta$ on the bilattice of belief pairs and argued that all major semantics for default logic can be derived from it. Among them are the semantics of weak extensions (Marek and Truszczyński, 1989a), partial weak extensions and the corresponding Kripke-Kleene semantics for default logic. In addition, the complete stable operator for $\mathcal{E}_\Delta$ coincides with the Guerreiro-Casanova operator characterizing extensions (Guerreiro and Casanova, 1990) and the $\leq_i$-least fixpoint of the stable operator $\mathcal{C}_{\mathcal{E}_\Delta}$ for $\mathcal{E}_\Delta$ yields the well-founded semantics for default logic described by Baral and Subrahmanian in (Baral and Subrahmanian, 1991). The semantics landscape of default logic is depicted in Figure 1.3.

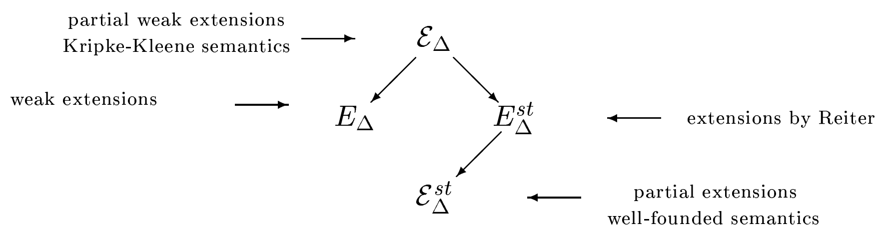

Figure 1.3 Operators and semantics associated with default logic

The similarity between the families of the semantics for logic programming, default logic and autoepistemic logic is striking. It has been long known that logic program clauses can be interpreted as default rules (Marek and Truszczyński, 1989b; Bidoit and Froidevaux, 1991). Namely, a logic program clause

$$a \leftarrow b_1, \ldots, b_m, \mathbf{not}(c_1), \ldots, \mathbf{not}(c_n)$$

can be interpreted as a default

$$\frac{b_1 \wedge \ldots \wedge b_m : \neg c_1, \ldots, \neg c_n}{a}$$

It turns out that under this translation the operators $\mathcal{T}_P$ and $\mathcal{E}_{\Delta(P)}$ are very closely related ($\Delta(P)$ stands for the default theory obtained from the logic program P by means of the translation given above). Namely, let us observe that each interpretation I can be associated with the possible-world structure consisting of all interpretations J such that $I(p) = \mathbf{t}$ implies $J(p) = \mathbf{t}$. Thus, the lattice $\mathcal{A}_2$ can be viewed as a sublattice of $\mathcal{W}$ and the restriction of the operator $\mathcal{E}_{\Delta(P)}$ to this sublattice essentially coincides with $\mathcal{T}_P$. It follows that all the derived operators are similarly related, and we obtain a perfect match between the semantics for logic programming and the semantics for default logic.

Similarly, in (Konolige, 1988) it was proposed to interpret a default

$$\frac{\beta_1 \wedge \ldots \wedge \beta_m : \neg\gamma_1, \ldots, \neg\gamma_n}{\alpha}$$

as a modal formula

$$K\beta_1 \wedge \ldots K\beta_m \wedge \neg K\neg\gamma_1 \wedge \ldots \wedge \neg K\neg\gamma_n \supset \alpha.$$

It turns out that under this translations the operators $\mathcal{E}_\Delta$ and $\mathcal{D}_{T(\Delta)}$ coincide (here $T(\Delta)$ is the modal image of a default theory Δ under

Konolige's translation). As before, all corresponding pairs of derived operators also coincide. Thus, we obtain a perfect match between the semantics for default and autoepistemic theories[2].

6. CONCLUSIONS

In the paper we presented an algebraic theory of fixpoints of non-monotone operators. We argued that essentially all major semantics for logic programming, autoepistemic logic and default logic can be described in an elegant and uniform way by applying our algebraic fixpoint theory to a particular operator: $\mathcal{T}_P$ in logic programming, $\mathcal{D}_T$ in autoepistemic logic, and $\mathcal{E}_\Delta$ in default logic. When, as our study appears to indicate, a number of different logics, developed from different perspectives, can be derived from a uniform principle, the question must be raised of the knowledge theoretic role and meaning of this principle.

We hypothesize that our theory provides a generalized algebraic account of non-monotone constructions and non-monotone induction in mathematics. Tarski's fixpoint theory can be considered as a general method for modeling monotone constructions and positive inductive definitions. It seems that the theory presented here extends this theory to the general case of non-monotone inductive definitions. The investigation of this hypothesis amounts to an empirical study of constructive techniques in mathematics and of logical formalizations of such techniques, including existing formalizations of non-monotone induction such as iterated inductive definitions and inflationary fixpoint logic. Early results in this direction are presented in (Denecker, 1998).

If we can validate our hypothesis, then the theory presented here elucidates new fundamental relationships between different scientific domains, including nonmonotonic reasoning, logic programming, database theory and inductive definitions. It may also shed more light on the role of different logics for knowledge representation. The discussion of these issues will be the subject of another publication.

Acknowledgments

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[2]However, this correspondence does not align expansions by Moore and extensions by Reiter. These two semantics occupy different locations in the corresponding hierarchies. A more detailed discussion of this issue can be found in (Denecker et al., 2000).

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